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Computing the Canonical Portalgon

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Abstract

A surface can be defined without reference to $\mathbb{R}^3$ from a *portalgon*, a collection of plane polygons (*fragments*) whose sides are partially matched, as described recently by Löffler, Ophelders, Staals, and Silveira (SoCG'23). The computation of shortest paths on a surface is affected by the maximum number of times they visit the image of a fragment, the *happiness* of the portalgon, which is unbounded, in stark contrast to polyhedral meshes in $\mathbb{R}^3$. While it is known that every surface admits portalgons of bounded happiness, efficiently computing one is open.

In this paper we introduce the *canonical* portalgon of a (closed) surface (obtained essentially by cutting the surface along the Delaunay tessellation of the points of non-zero curvature), and we provide an algorithm to compute it from any other (triangulated) portalgon of the surface (polynomial in the number of fragments, and in the logarithm of the maximum aspect ratio of the fragments). This portalgon (after triangulating fragments for degenerate inputs) has bounded happiness by a result of Löffler, Ophelders, Staals, and Silveira. This implies algorithms to pre-process a portalgon before computing shortest paths on its surface, and to determine if the surfaces of two portalgons are isometric.

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1 Introduction

In one of its simplest forms a surface is a point set equipped with a distance function, or metric. Surfaces are often obtained from *polyhedral meshes*, straight polygons in $\mathbb{R}^3$ glued along their edges. The (intrinsic) distance between two points of the mesh is the length of a shortest path between them along the mesh. A surface can also be defined without reference to $\mathbb{R}^3$ from a *portalgon*, a collection of plane polygons (*fragments*) with matched sides. This very simple model is more general than polyhedral meshes. Recently Löffler, Ophelders, Staals, and Silveira [12] (see also [18]) proposed to unify the problems of polyhedral meshes that can be expressed without reference to $\mathbb{R}^3$ (are intrinsic) within the framework of portalgons.

Not all portalgons are suitable for computation. Prominently, shortest path algorithms are affected by the *happiness* of the portalgon, the maximum number of times the shortest paths of its surface visit the image of a fragment, which is unbounded (a fact noted almost 20 years ago in a popular blog post by Erickson [6]), in stark contrast with polyhedral meshes (whose edges are shortest paths in their surface). While every surface admits portalgons of bounded happiness, efficiently computing one is open.

The contribution of this paper is threefold. First we introduce the *canonical* portalgon of a (closed) surface (obtained essentially by cutting the surface along the Delaunay tessellation of the points of non-zero curvature). This portalgon (after triangulating fragments for degenerate inputs) has bounded happiness by a result of [12]. Second and most importantly, we provide an efficient algorithm to compute the canonical portalgon from any other portalgon of the surface. Last but not least, our algorithm directly applies to pre-process a portalgon before computing shortest paths on its surface, and to determine if the surfaces of two portalgons are isometric.

Before describing our results in more detail, we survey related works.

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1.1 Related works

Some shortest path algorithms operate in the plane [14, 8, 20]. Other algorithms are designed specifically for polyhedral meshes. Mitchel, Mount, and Papadimitriou [15] compute single-source shortest paths, and Mount [16] computes a Voronoi diagram of the input surface S, a decomposition of the points of S according to which source(s) they are closer to (see also [3, 10, 11]). Both algorithms run in time polynomial in the number of sources and in the number of edges of the mesh. Roughly, they propagate waves along the surface, starting from the source(s).

Löffler, Ophelders, Staals, and Silveira adapt the single-source shortest paths algorithm [12, Section 3] to portalgons, whose running time now depends on the happiness of h the portalgon. They prove that cutting a surface along a Delaunay triangulation would provide a portalgon of bounded happiness [12, Section 4], but observe that no efficient algorithm is known to compute them. They compute portalgons of bounded happiness [12, Section 5], but only for a restricted class of inputs whose surfaces are all homeomorphic to an annulus.

For comparing surfaces the only algorithms we are aware of are heuristic [2, 5, 13, 17].

1.2 Our results

1.2.1 Main result

It is classical (and detailed in Appendix G) that if a closed surface S is not flat, then the Voronoi diagram of the points of non-zero curvature is dual to a Delaunay tessellation D of S (here the 1-skeleton of the Voronoi diagram is the set of points whose distance to the sources is realized by several shortest paths; in particular the open Voronoi cells are homeomorphic to disks). We define the *canonical* portalgon of S as the one obtained by cutting S open along D. If S is flat, and thus homeomorphic to a torus by the Gauss-Bonnet formula, we consider the Voronoi diagram of a single arbitrary source, as the resulting portalgon does not depend on the source by symmetry of S.

A portalgon P is *triangular* if all its fragments are triangles. The *global* aspect ratio of P is then the greatest side length of a fragment of P divided by the smallest height of a fragment of P. Our main result is that if the surface $\mathcal{S}(P)$ of P is closed, then the canonical portalgon of $\mathcal{S}(P)$ can be computed from P in time polynomial in the number n of fragments of P and in the *logarithm* of the global aspect ratio of P:

▶ **Theorem 1.** *Let P be a triangular portalgon, with n fragments, of global aspect ratio r, whose surface $\mathcal{S}(P)$ is closed. One can compute the canonical portalgon of $\mathcal{S}(P)$ in $O^*(n^3 \log^4 r)$ time.*

Here and in the rest of the paper $O^*()$ stands for for domination up to a poly-logarithmic factor. Also $\log(\cdot)$ denotes $\log_2(\lceil\cdot\rceil) + 1$. We analyze all our results in the real RAM model of computation. Let us mention that all our algorithms remain polynomial in the number of fragments and in the logarithm of the aspect ratio when measured in terms of the *local* aspect ratio of the input portalgon P, the maximum aspect ratio of its fragments, where the aspect ratio of a fragment is its maximum side length divided by its smallest height. Indeed global and local aspect ratios are related by the following, proved in Appendix A:

▶ **Lemma 2.** *Let P be a triangular portalgon, with n fragments, whose global and local aspect ratios are respectively r and r', whose surface $\mathcal{S}(P)$ is connected. Then $r' \le r \le (r')^n$.*

The aspect ratios of a triangular portalgon P are natural parameters that can be read off from P. On the other hand there is no known algorithm to compute the happiness of P.

1.2.2 Applications

We now mention two immediate but important consequences of our Theorem 1. First, by a result of [12, Section 4], cutting the fragments of our canonical portalgon into triangles along arcs provides a portalgon of bounded happiness. Combined with Theorem 1 this gives the following, proved in Appendix A:

▶ **Corollary 3.** *Let P be a triangular portalgon, with n fragments, of global aspect ratio r, whose surface $\mathcal{S}(P)$ is closed. One can compute in $O^*(n^3 \log^4 r)$ time a triangular portalgon P' of $\mathcal{S}(P)$ that has $O(n)$ fragments and bounded happiness.*

On the output portalgon P' the single-source shortest path algorithm of [12, Section 3] would run in time $O^*(n^2)$. Second, one can determine if two surfaces are isometric by testing if they have the same canonical portalgon (with a labeled combinatorial map isomorphism test). This gives the following, proved in Appendix A:

▶ **Corollary 4.** *Let P and P' be triangular portalgons, with at most n fragments, of global aspect ratios smaller than or equal to r, whose surfaces $\mathcal{S}(P)$ and $\mathcal{S}(P')$ are closed. One can determine if $\mathcal{S}(P)$ and $\mathcal{S}(P')$ are isometric in $O^*(n^3 \log^4 r)$ time.*

1.3 Overview and techniques for the proof of Theorem 1

On the surface $\mathcal{S}(P)$ of a portalgon P, we consider the graph $\mathcal{C}(P)$ traced by the sides of the fragments of P. Every edge e of $\mathcal{C}(P)$ is a *segment* of $\mathcal{S}(P)$, a geodesic relatively disjoint from the curved points. Adapting the notion of happiness to our needs, we define the *segment-happiness* of e as the maximum number of times it is visited by a shortest path. The maximum segment-happiness of the edges of $\mathcal{C}(P)$ then defines the segment-happiness of P. On triangular portalgons, happiness and segment-happiness are equivalent up to a constant factor.

To prove Theorem 1 we first consider portalgons P whose surface $\mathcal{S}(P)$ is simply connected and has no positively curved point in its interior. Indeed the Gauss-Bonnet formula implies that every segment of $\mathcal{S}(P)$ is the unique geodesic between its two endpoints, and thus the unique shortest path, so no shortest path crosses it twice. In trying to leverage this key property, we consider a wider class of surfaces by dispensing ourself from the constraint on topology, but keeping the constraint on curvature. More precisely we consider (connected) surfaces $\mathcal{S}(P)$ that are not simply connected and have no positively curved point in their interior. The *systole* of $\mathcal{S}(P)$ is the smallest length of a non-contractible geodesic closed curve in $\mathcal{S}(P)$. Our key technical result toward the proof of Theorem 1 is:

▶ **Proposition 5.** *Let P be a triangular portalgon with n fragments, of maximum fragment edge length L. Assume that the surface $\mathcal{S}(P)$ of P is connected, is not simply connected, and has no positively curved point in its interior. Let $s > 0$ be at most the systole of $\mathcal{S}(P)$. One can compute in $O(n \log^2(n) \log^2(L/s))$ time a triangular portalgon of $\mathcal{S}(P)$ with $O(n \log(L/s))$ fragments, and of segment-happiness $O(\log(n) \log^2(L/s))$.*

Note that in Proposition 5 the surface $\mathcal{S}(P)$ may have boundary, in which case the algorithm maintains the correspondence between the boundary components of the input and those of the output. The algorithm for Proposition 5 uses four elementary operations on the portalgon P that we describe in Section 3. As a tool to analyze the algorithm we introduce in Section 4 a new parameter on the segments of $\mathcal{S}(P)$, the *enclosure*, possibly of independent interest, that dominates segment-happiness well enough to our needs. In the same section we show what the elementary operations do to the enclosure and the length of the edges of $\mathcal{C}(P)$. In Section 5 we finally describe and analyze the algorithm proving Proposition 5. Essentially, we produce a triangulation of $\mathcal{S}(P)$ decomposed into a central region whose edges have low enclosure, and into a set of tubular regions whose edges may have high enclosure but have low segment-happiness anyway.

At the end of Section 5 we extend Proposition 5 to surfaces having positively curved points, essentially by cutting out caps around those points. Also we deduce the canonical portalgon from the Voronoi diagram of its vertices, that we obtain by adapting the shortest path algorithm of [12, Section 3]. Immediately, we deduce Theorem 1.

2 Preliminaries

We assume basic knowledge of topology of surfaces and covering spaces; see, e.g., Stillwell [19].

2.1 Portalgons, surfaces, and isometry

Let X be a set. A ***metric*** on X is a map $d : X^2 \to \mathbb{R}$, that is symmetric, positive on distinct elements, null on equal elements, and satisfies the triangular inequality. Then (X, d) is a ***metric space***. An isometry is a one-to-one correspondence $f : X \to X'$ between two metric spaces (X, d) and (X', d') such that $d(x, y) = d'(f(x), f(y))$ for every $x, y \in X$. Two metric spaces are ***isometric*** if there exists an isometry between them.

We call ***polygon*** any finite cycle Q embedded in the Euclidean plane by straight line segments. Two polygons are considered equal if one can be obtained from the other by translations and rotations. The compact region of the plane bounded by Q is a metric space, the ***surface*** of Q. A ***portalgon*** P is a set of polygons, the ***fragments*** of P, along with a partial matching of the fragment edges, such that every two matched edges have the same length. Every subset of the fragments of P induces a ***sub-portalgon*** P' of P, where two fragment edges are matched in P' if and only if they are matched in P. A ***triangle*** is a polygon with three vertices. A portalgon is ***triangular*** if all its fragments are triangles.

Given a portalgon P, realize the surfaces of the fragments of P disjointly. In their union identify every two matched edges with an orientation-preserving isometry. The result is a metric space $\mathcal{S}(P)$, the ***surface*** of P. The distance between two points of $\mathcal{S}(P)$ is the smallest length of a path between them in $\mathcal{S}(P)$, where the length of a path is measured in the fragments of P. Note that $\mathcal{S}(P)$ is orientable, and may have boundary. More generally we call ***surface*** any metric space S isometric to the surface of a portalgon. And when we say that P is a portalgon of S, we identify $\mathcal{S}(P)$ and S with an isometry. The sides of the fragments of P map to a graph $\mathcal{C}(P)$ in $\mathcal{S}(P)$, the ***carrier*** of P. A ***face*** of $\mathcal{C}(P)$ is a connected component of $\mathcal{S}(P) \setminus \mathcal{C}(P)$. If P is triangular then $\mathcal{C}(P)$ is a ***triangulation***.

2.2 Curvature and geodesics

In a surface S a point x is ***flat*** if there is a neighborhood of x isometric to a plane metric disk, or half-disk, otherwise x is ***curved***. If P is a portalgon of S, then every curved point x of S is a vertex of $\mathcal{C}(P)$. The sum a of the angles of the corners of faces of $\mathcal{C}(P)$ around x does not depend on P. If x lies in the interior of S then either $a < 2\pi$ or $a > 2\pi$, and we say that x is ***positively*** or ***negatively*** curved respectively.

In this paper we denote by $\ell(p)$ the length of a path p. A ***geodesic*** is a path p in S whose relative interior is locally straight outside of the curved points of S, and does the following at each curved point x of S. If x lies in the interior of S, then p forms at x an angle greater than or equal to π on both sides (then x is negatively curved). Otherwise p

forms an angle greater than or equal to π on the side that does not contain the boundary of S. Equivalently, geodesics are paths that are locally shortest. A ***segment*** is a geodesic relatively disjoint from the curved points of S.

We now focus on a surface S that has no positively curved point in its interior. The Gauss-Bonnet formula, applied to the universal covering space of S, implies that for every path p in S there is a unique geodesic path p' homotopic to p in S, and $\ell(p') \leq \ell(p)$. If moreover S is not simply connected, then the ***systole*** of S is the smallest length of a non-contractible geodesic closed curve in S (note that this definition of takes into account the curves homotopic to a boundary component of S). Importantly, every segment shorter than half the systole of S is then the unique shortest path between its endpoints.

2.3 Happiness

Let S be a surface. Löffler, Ophelders, Staals, and Silveira [12] define the ***happiness*** of a portalgon P of S as the maximum number of times a shortest path on S visits the image of a single fragment of P. Adapting this notion to our needs, we define the ***segment-happiness*** $h_S(e)$ of a segment e of S as the maximum number of intersections between e and a shortest path of S. The ***segment-happiness*** of P is then the maximum $h_S(e)$ over the edges e of $\mathcal{C}(P)$. If P is triangular then its happiness and segment-happiness are equivalent up to a constant factor, and segment-happiness suits better the analysis of our algorithms.

2.4 Tubes and bifaces

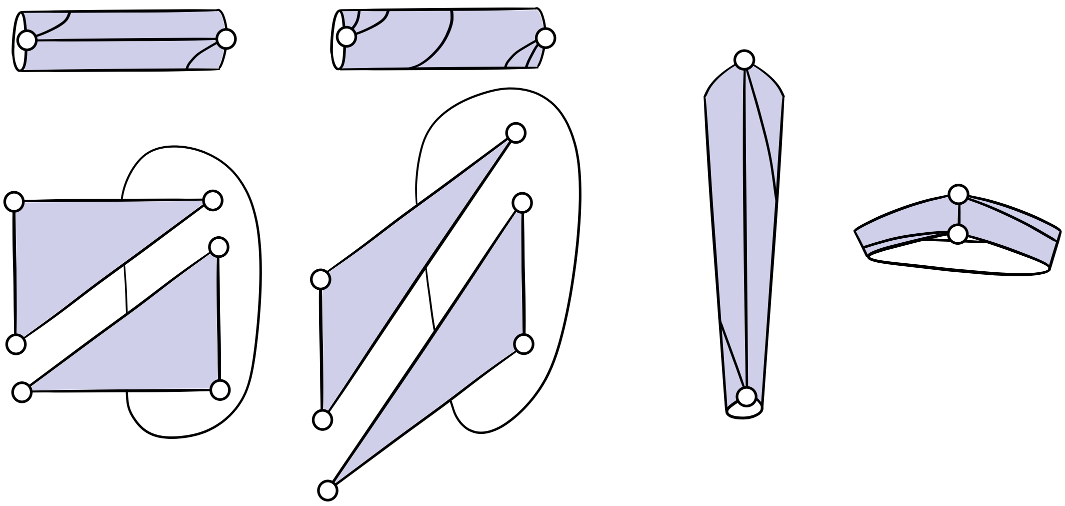

Figure 1 (From left to right) A good biface, a biface not good, a thin biface, a thick biface.

In this section we focus on a particular class of partalgons, similarly to [12, Section 5]. See Figure 1. A ***tube*** is a triangular portalgon X whose surface $\mathcal{S}(X)$ is homeomorphic to an annulus, has no curved point in its interior, and such that $\mathcal{C}(X)$ has one vertex per boundary component of $\mathcal{S}(X)$. A ***biface*** is a triangular portalgon B with two fragments whose respective edges e_0, e_1, e_2 and e'_0, e'_1, e'_2, in order, are such that e_0 is matched with e'_0 and e_1 is matched with e'_1. Then $\mathcal{C}(B)$ has four edges, two loop edges forming the two boundary components of $\mathcal{S}(B)$, that we call ***boundary edges***, and two ***interior edges*** relatively included in the interior of $\mathcal{S}(B)$. We say that B is ***good*** if the two interior edges

e and f of $\mathcal{C}(B)$ are such that e is a shortest path in $\mathcal{S}(B)$, and f is a shortest arc of the (possibly non-convex) quadrilateron obtained by cutting $\mathcal{S}(B)$ along e. While tubes and bifaces have unbounded happiness, good bifaces on the other hand are designed to satisfy the following, whose straightforward proof is detailed in Appendix B:

▶ **Lemma 6.** *If e is an interior edge of a good biface B, then $h_{\mathcal{S}(B)}(e) < 7$.*

We will distinguish good bifaces by saying that a good biface B is ***thin*** if every interior edge of $\mathcal{C}(B)$ is longer than every boundary edge of $\mathcal{C}(B)$, and that B is ***thick*** otherwise.

3 The elementary operations

In this section we describe, on a portalgon P, the four elementary operations that will be used by the algorithm of Proposition 5. See Figure 2.

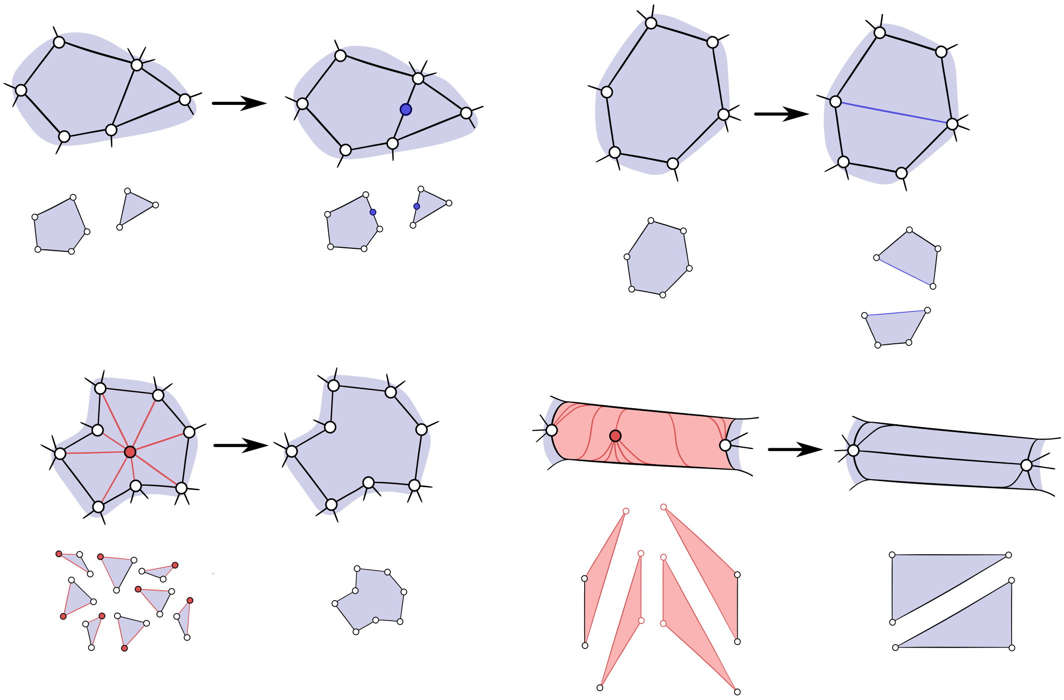

Figure 2 The four elementary operations used by the algorithm of Proposition 5.

1 Inserting a vertex in an edge. Given an edge e of $\mathcal{C}(P)$, one can insert a point in the relative interior of e as a vertex in $\mathcal{C}(P)$ by inserting a vertex in each fragment edge of P corresponding to e.

2 Inserting an arc in a face. Consider a face F of $\mathcal{C}(P)$. An ***arc*** a of F is a geodesic path in $\mathcal{S}(P)$ whose relative interior is included in F and whose end-points are vertices of $\mathcal{C}(P)$. One can insert a as an edge in $\mathcal{C}(P)$ by cutting the fragment of P corresponding to F in two.

3 Deleting a vertex. Assume that $\mathcal{C}(P)$ is a triangulation (equivalently, that P is triangular), and consider a vertex v of $\mathcal{C}(P)$ that lies in the interior of $\mathcal{S}(P)$, is flat, and is not incident to any loop edge in $\mathcal{C}(P)$. (In particular v does not occur twice in a fragment of P.) One can delete v and its incident edges from $\mathcal{C}(P)$ by merging the fragments of P in which v occurs into a single fragment. Note that then $\mathcal{C}(P)$ is not a triangulation anymore.

4 Replacing a tube by a good biface. Given a sub-portalgon X of P (Section 2.1), if X is a tube (Section 2.4), then we consider the elementary operation of replacing X by a good biface in P. In [12, Theorem 45] (building upon a ray shooting algorithm of [7]) they provide an algorithm to transform a biface into a triangular portalgon with bounded number of fragments and bounded happiness. While this algorithm extends from bifaces to triangular portalgons X such that the dual graph of $\mathcal{C}(X)$ in $\mathcal{S}(X)$ has at most one simple cycle, it does not immediately extend to tubes. In Appendix C we use their result to prove the following:

▶ **Lemma 7.** *Let X be a tube with n fragments, of maximum fragment edge length L. Let $s > 0$ be at most the systole of $\mathcal{S}(X)$. One can compute in $O(n \log(n) \log(L/s))$ time a good biface of $\mathcal{S}(X)$.*

In a nutshell, the algorithm of Lemma 7 greedily deletes vertices of $\mathcal{C}(X)$, and inserts arcs in the resulting faces to make $\mathcal{C}(X)$ a triangulation again, until every vertex of $\mathcal{C}(X)$ is incident to a loop edge, at which point X is a concatenation of bifaces. Then it replaces every biface by a good biface using [12, Theorem 45]. Finally it repeatedly merges pairs of adjacent good bifaces into a single good biface.

4 Enclosure

In this section we fix a surface of the kind of the input of Proposition 5. More precisely a compact surface S, connected, not simply connected, such that the interior of S does not contain any positively curved point. We introduce a parameter on the segments of S that we call *enclosure*. Then we relate enclosure to segment-happiness and length, and we show what the elementary operations of Section 3 do to the enclosure and the length of the edges involved, preparing the analysis of the algorithm of Proposition 5 in Section 5.

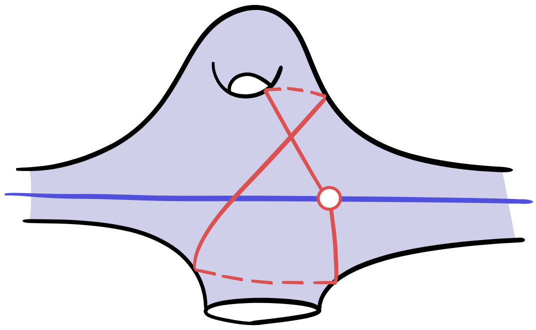

Figure 3 The red loop encloses the blue segment in the surface.

First we define enclosure. See Figure 3. Let e be a segment in S. If x is a point in the relative interior of e then $\langle x\rangle_e$ denotes the minimum length of the two sub-segments of e separated by x. Let γ be a loop based at x in S, geodesic except possibly at its basepoint. Assume that $\ell(\gamma) < \langle x\rangle_e$. Then γ is in general position with e (γ and e do not overlap). If moreover γ meets x on *both* sides of e (so that in particular e is not included in the boundary of S), then we say that γ ***encloses*** e ***in*** S. Also we say that γ encloses e ***by a factor of*** $\langle x\rangle_e/\ell(\gamma)$ in S. The ***enclosure*** $c_S(e)$ is the supremum of the ratios $\langle x\rangle_e/\ell(\gamma)$ over the loops γ enclosing e in S, conventionally set to one if there is no loop enclosing e in S.

We will use the following propositions about enclosure, all proved in Appendix D. The following relates enclosure to segment-happiness and length:

▶ **Proposition 8.** *Let e be a segment of S. Let $s > 0$ be at most the systole of S. Assume that there exists a triangular portalgon of S with n fragments, and of maximum fragment edge length L. Then $h_S(e) \leq 600 \cdot c_S(e) \cdot (\log(c_S(e)) + \log(n) + \log(L/s))$ and $\ell(e)/s \leq 600 \cdot c_S(e) \cdot n \cdot \lceil L/s \rceil^2$.*

Given a portalgon P of S, we may insert a point in the relative interior of an edge of $\mathcal{C}(P)$ as a vertex in $\mathcal{C}(P)$, with the elementary operation 1 (Section 3). The following enforces that the two resulting edges are not more enclosed in S than the initial edge. It is straightforward:

▶ **Lemma 9.** *Let e be a segment in S, and let f be a sub-segment of e. Then $c_S(e) \geq c_S(f)$.*

Consider a face F of $\mathcal{C}(P)$. Among all the arcs of F consider the *shortest* one(s). We may insert such a *shortest* arc of F as an edge in $\mathcal{C}(P)$ with the elementary operation 2 (Section 3). The following enforces that if the arc inserted is«very enclosed»in S, then it is«not much more enclosed»in S and«not much longer»than the edges initially in $\mathcal{C}(P)$:

▶ **Proposition 10.** *Let F be a face of the carrier of a portalgon of S. Assume that F has a shortest arc e such that $c_S(e) > 6$. Then F has a boundary edge f such that $c_S(f) \geq c_S(e) - 4$ and $\ell(f) \geq (1 - 4/c_S(e))\ell(e)$.*

We may replace a sub-portalgon of P by a good biface B with the elementary operation 4 (Section 3). The following enforces that if B is thick, and if an interior edge of $\mathcal{C}(B)$ is«very enclosed»in S, then it is«not much more enclosed»in S and«not much longer»than the edges initially in $\mathcal{C}(P)$, similarly to Proposition 10:

▶ **Proposition 11.** *Assume that a portalgon of S admits a thick biface B as a sub-portalgon, and let e be one of the two interior edges of $\mathcal{C}(B)$. Assume that $c_S(e) > 6$. Then there is a boundary edge f of $\mathcal{C}(B)$ such that $c_S(f) \geq c_S(e) - 5$ and $\ell(f) \geq (1 - 4/c_S(e))\ell(e)$.*

We will keep the thin bifaces encountered in the output portalgon. The following enforces that their boundary edges are«not very enclosed»in S:

▶ **Proposition 12.** *Assume that a portalgon of S admits a thin biface B as a sub-portalgon, and let e be a boundary edge of $\mathcal{C}(B)$. Then $c_S(e) \leq 2$.*

5 Computing happier portalgons on non-positively curved surfaces

In this section we prove Proposition 5 with an algorithm that uses the elementary operations of Section 3, that we analyze with the properties of Section 4. First we describe the algorithm in Section 5.1. We proceed with the analysis in the other sections, and we finally prove Proposition 5 in Section 5.5.

As in Proposition 5 we fix an input triangular portalgon P with n fragments, of maximum fragment edge length L, whose surface $S := \mathcal{S}(P)$ is connected, is not simply connected, and has no positively curved point in its interior. We let $s > 0$ be at most the systole of S.

5.1 Algorithm

The data structure maintains a portalgon R of S, and the following decomposition of R. See Figure 4. The fragments of R are partitioned into a sub-portalgon R_A of R, the ***active region*** (the surface of R_A may be disconnected), and into other sub-portalgons of R, the

inactive regions. The first invariant maintained by the algorithm is that every inactive region is a good biface.

The pairs of fragment edges that lie in different regions and are matched in R form a set of edges in the interior of S, all loops by the first invariant, that we call ***inactive loops*** (some inactive loops may not be incident to the surface of R_A). The second invariant maintained by the algorithm is that the inactive loops are pairwise-disjoint (equivalently, no two of them are based at the same vertex).

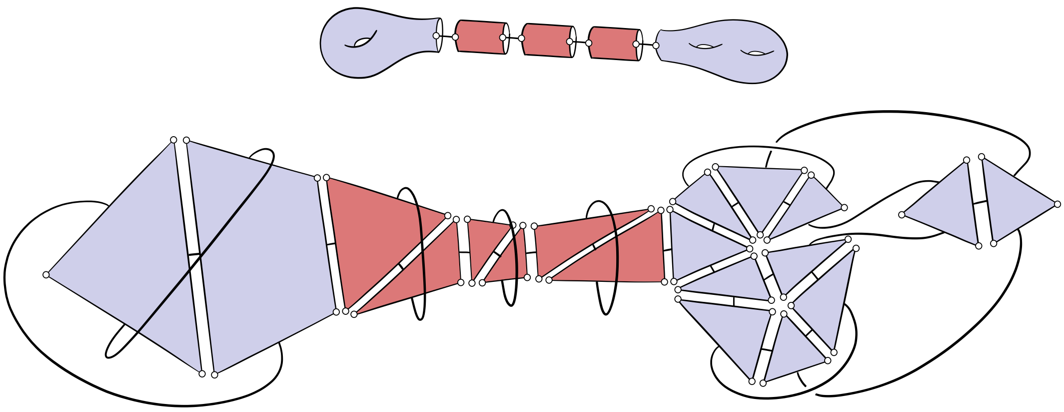

Figure 4 The data-structure of Section 5: a portalgon R decomposed into an active region R_A (in blue) and some inactive regions (in red).

The algorithm calls three routines that we detail below. It involves a positive integer constant κ, and is correct whenever $\kappa \geq 326$, as we shall see. Yet we leave κ as an indeterminate for now in order to clarify the analysis. The algorithm is the following:

- Initialize the active region R_A as the input portalgon P (without any inactive region).
- Repeat $\log(L/s)$ times the following:
  - Apply SUBDIVISION.
  - Repeat κ times the following: apply TUBING then DELETION.

Importantly R_A is triangular in-between routines, but usually not inside a routine.

SUBDIVISION. Consider every edge e of $\mathcal{C}(R_A)$ that is not included in the boundary of $\mathcal{S}(R_A)$, and insert the middle point of e as a vertex in $\mathcal{C}(R_A)$. Then insert shortest arcs in the faces of $\mathcal{C}(R_A)$, in any order, as long as possible, making $\mathcal{C}(R_A)$ a triangulation again.

DELETION. Consider the vertices of $\mathcal{C}(R_A)$ that lie in the interior of $\mathcal{S}(R_A)$, are flat, have degree smaller than or equal to six, and are not incident to any loop edge. Delete a maximal (though not necessarily maximum) independent set of such vertices from $\mathcal{C}(R_A)$. Then insert shortest arcs in the resulting faces of $\mathcal{C}(R_A)$, in any order, as long as possible, making $\mathcal{C}(R_A)$ a triangulation again.

TUBING. This last routine is slightly more technical, and is in three steps:

1. Consider every connected component of $\mathcal{S}(R_A)$ whose corresponding sub-portalgon X of R_A is a tube. Replace X by a good biface B. Remove B from R_A.
2. Build a set J of loop edges of $\mathcal{C}(R_A)$ that lie in the interior of $\mathcal{S}(R_A)$ and are pairwise-disjoint, as follows. There are two cases:

   a. If $\mathcal{S}(R_A)$ is a flat torus (and thus $R_A = R$, since S is connected), do the following. Let J contain two disjoint loop edges of $\mathcal{C}(R_A)$ if there are any, otherwise let $J = \emptyset$.
   b. Otherwise, if $\mathcal{S}(R_A)$ is not a flat torus, do the following. First construct a set J' of loop edges by considering every vertex v of $\mathcal{C}(R_A)$ that lies in the interior of $\mathcal{S}(R_A)$ and is incident to a loop edge, and by putting one (and only one) of the loop edges incident to v in J'. Then build $J \subseteq J'$ by removing from J' every $e \in J'$ that satisfies each of the following: the vertex of e is flat, there are two distinct connected components of $\mathcal{S}(R_A) \setminus J'$ adjacent to e, say S_0 and S_1, and the two sub-portalgons of R_A corresponding to S_0 and S_1 are both tubes.

3. Consider every connected component of $\mathcal{S}(R_A) \setminus J$ whose corresponding sub-portalgon X of R_A is a tube. Replace X by a good biface B. If B is thin remove B from R_A.

The idea behind step 2 is to remove loops from J so that step 3 simplifies a concatenation of tubes into a single good biface when possible, instead of simplifying the tubes separately into several good bifaces.

5.2 The inactive loops are not very enclosed

We now begin the analysis of the algorithm. First we prove that at any time during the execution the inactive loops are«not very enclosed»in S:

▶ **Lemma 13.** *At any time during the execution of the algorithm, every inactive loop e satisfies $c_S(e) \leq 2$.*

Proof. Only the third step of TUBING may create an inactive loop, by removing a *thin* biface B from R_A. Then the routines cannot modify B. So the algorithm maintains the invariant that every inactive loop e is adjacent to the surface of at least one inactive region that is a *thin* biface, and thus that $c_S(e) \leq 2$ by Proposition 12. ◀

5.3 The geometry of the active region is simplified

In this section we show that running the algorithm simplifies the geometry of the active region R_A. More precisely the maximum length of the edges of $\mathcal{C}(R_A)$ that are«very enclosed»in S (if any) scales down exponentially. Recall that L denotes the maximum fragment edge length of the *input* portalgon P:

▶ **Proposition 14.** *After $i \geq 1$ iterations of the main loop, let e be an edge of $\mathcal{C}(R_A)$. If $c_S(e) \geq 60i\kappa$ then $\ell(e) < 2^{1-i}L$.*

Roughly, the reason is that all those edges are cut in two by the SUBDIVISION routine at the beginning of the main loop, and that the rest of the main loop does not insert in $\mathcal{C}(R_A)$ edges that are both«very enclosed»and«much longer»than the edges already in $\mathcal{C}(R_A)$.

Note that this is why step 3 of TUBING must remove from R_A every thin biface B encountered: the interior edges of $\mathcal{C}(B)$ may be«very enclosed»in S and«much longer»than the edges already in $\mathcal{C}(R_A)$.

We sketch the proof of Proposition 14, and defer the complete proof to Appendix E.

Sketch of proof. We have three claims, one for each routine. Consider the value of R at some point in the execution of the algorithm. Let R' result from applying SUBDIVISION to R, and let e' be an edge of $\mathcal{C}(R'_A)$. Our first claim is that if $c_S(e') > 14$, then there is an edge e in $\mathcal{C}(R_A)$ such that $c_S(e) \geq c_S(e') - 12$ and $\ell(e) \geq 2(1 - 12/c_S(e'))\ell(e')$. Let us prove this claim. First, observe that e' is not included in the boundary of $\mathcal{S}(R'_A)$, since e' is enclosed and thus not included in the boundary of S, and since e' is not an inactive loop by Lemma 13. Second, the routine starts by inserting the middle point of some edges e of $\mathcal{C}(R_A)$ as a vertex in $\mathcal{C}(R_A)$. If e' is one of the resulting two half-segments of e, then $\ell(e) = 2\ell(e')$ and $c_S(e) \geq c_S(e')$ by Lemma 9. Finally, given a face F of $\mathcal{C}(R_A)$, the fragment Q of R_A corresponding to F is a triangle before the routine. Then the routine may insert the middle point of some of the boundary edges of Q as vertices of Q, so Q may have up to six vertices during the routine, and so the routine may insert up to three arcs in F, cutting Q into at most four fragments. If e' is one of the arcs inserted in F, Proposition 10 applied at most 3 times implies that there is a boundary edge f of F such that $c_S(f) \geq c_S(e') - 12$ and $\ell(f) \geq (1 - 12/c_S(e'))\ell(e')$. By the preceding f is not included in the boundary of $\mathcal{S}(R_A)$, so f is a half-segment of an edge e of $\mathcal{C}(R_A)$, $\ell(e) = 2\ell(f)$, and $c_S(e) \geq c_S(f)$. That proves the first claim.

The complete proofs of the second and third claims are deferred to Appendix E. Our second claim is that if R' results from applying DELETION to R, and if $c_S(e') > 13$, then there is an edge e in $\mathcal{C}(R_A)$ such that $c_S(e) \geq c_S(e') - 12$ and $\ell(e) \geq (1 - 12/c_S(e'))\ell(e')$. The reason is that if e' does not initially belong to $\mathcal{C}(R_A)$ then e' is an arc inserted by the routine, and Proposition 10 applies. Our third claim is that if R' results from applying TUBING to R, and if $c_S(e') > 6$, then there is an edge e in $\mathcal{C}(R_A)$ such that $c_S(e) \geq c_S(e') - 5$ and $\ell(e) \geq (1 - 4/c_S(e'))\ell(e')$. The reason is that if e' does not initially belong to $\mathcal{C}(R_A)$ then e' is an interior edge of a thick biface placed by the routine, and Proposition 11 applies.

Finally we prove the proposition. Let $R = P$ be the input portalgon. Let R' result from applying i iterations of the main loop to R, and assume that there is an edge e' in $\mathcal{C}(R'_A)$ such that $c_S(e') \geq 60i\kappa$. DELETION and TUBING where applied $i\kappa$ times each, and SUBDIVISION i times. Also $(12 + 5)i\kappa + 12i \leq 29i\kappa$. So our three claims imply that there is an edge e in $\mathcal{C}(R_A)$ such that $\ell(e) \geq 2^i(1 - 29i\kappa/c_S(e'))\ell(e') > 2^{i-1}\ell(e')$. And $\ell(e) \leq L$ since e belongs to the input portalgon. ◀

5.4 The combinatorial size of the active region is bounded

In this section we show that when running the algorithm the number m_A of vertices of $\mathcal{C}(R_A)$ stays dominated by a linear function of the number n of fragments of the *input* portalgon P:

▶ **Proposition 15.** *There is a universal constant $\lambda > 0$ for which the following holds. Assume $\kappa \geq 326$. Let R' result from applying one iteration of the main loop to R. If $\mathcal{C}(R_A)$ has more than $\lambda \cdot n$ vertices, then $\mathcal{C}(R'_A)$ has less vertices than $\mathcal{C}(R_A)$.*

Roughly, the reason is that as long as m_A exceeds n by a constant factor, m_A is multiplied by at most a constant factor by SUBDIVISION at the beginning of the main loop, and m_A is divided by at least a constant factor by each application of TUBING and DELETION. By iterating TUBING and DELETION $\kappa \geq 326$ times we make sure that m_A is decreased by the main loop.

Note that DELETION is useless at deleting vertices from $\mathcal{C}(R_A)$ if most of the vertices of $\mathcal{C}(R_A)$ lie on the boundary of $\mathcal{S}(R_A)$, or if most of the vertices in the interior of $\mathcal{S}(R_A)$ are incident to a loop edge. Applying TUBING before DELETION ensures that this does not happen.

We need the three following lemmas, proved in Appendix E. Lemma 17 and Lemma 18 are straightforward consequences of Euler's formula. Lemma 17 is similar to [9, Lemma 3.2].

▶ **Lemma 16.** *Let Y be a triangular portalgon whose surface $\mathcal{S}(Y)$ is connected, has genus g, b boundary components, and c curved points in its interior. Let I be a set of loop edges of*

$\mathcal{C}(Y)$ that lie in the interior of $\mathcal{S}(Y)$ and are pairwise-disjoint. At most $9(g+b+c)$ loops in I are adjacent to only one connected component of $\mathcal{S}(Y) \setminus Y$, or are adjacent to a connected component of $\mathcal{S}(Y) \setminus I$ whose corresponding sub-portalgon of Y is not a tube.

▶ **Lemma 17.** *In a triangulation of genus g, with b boundary components, and with $m > 24(g+b)$ vertices, at least $m/12$ vertices have degree smaller than or equal to 6.*

▶ **Lemma 18.** *Every triangulation of genus g with m vertices has less than $6(g+m)$ edges.*

Proof of Proposition 15. Let $N > 9n$ so that N is greater than the number of vertices of the input portalgon P, and greater than the sum of the genus, the number of boundary components, and the number of curved points in the interior of S. If $\mathcal{C}(R_A)$ has $m_A > 6N$ vertices before SUBDIVISION, then $\mathcal{C}(R_A)$ has less than $7m_A$ vertices after SUBDIVISION. Indeed there are no more vertices inserted by SUBDIVISION than there are edges in $\mathcal{C}(R_A)$ initially, and there are less than $6(N+m_A)$ such edges by Lemma 18, since the genus of $\mathcal{S}(R_A)$ is smaller than or equal to the genus of S. Now consider one iteration of TUBING and DELETION. This iteration does not create any new vertex. We claim that if $\mathcal{C}(R_A)$ has $m_A > 744N$ vertices right after TUBING, then at this point the interior of $\mathcal{S}(R_A)$ contains more than $m_A/24$ flat vertices of degree smaller than or equal to six not incident to any loop edge. First we show why the claim implies the proposition. Any maximal independent set of such vertices contains at least $m_A/(24 \times 7) = m_A/168$ vertices. So DELETION removes at least $m_A/168$ vertices. It follows that if $\mathcal{C}(R_A)$ has $m_A > 749N$ vertices before the iteration of TUBING and DELETION, then $\mathcal{C}(R_A)$ has less than $167m_A/168$ vertices after the iteration. Indeed either $\mathcal{C}(R_A)$ already has at most $744N$ vertices right after TUBING, and $744/749 < 167/168$, or at most $167/168$ of the vertices in $\mathcal{C}(R_A)$ after TUBING remain in $\mathcal{C}(R_A)$ after DELETION. That implies the proposition as $7 < (168/167)^\kappa$ since $\kappa \geq 326$.

Now we prove the claim. First, observe that after the first step of TUBING the set I of inactive loops adjacent to $\mathcal{S}(R_A)$ contains less than $9N$ inactive loops. Indeed after the first step of TUBING, for every connected component S_0 of $\mathcal{S}(R_A)$, the sub-portalgon of R_A corresponding to S_0 is not a tube. So this follows from Lemma 16 applied to $\mathcal{C}(R)$ and I.

Second, observe that less than $10N$ loops are kept in J by the second step of TUBING. Indeed Lemma 16 applies to $\mathcal{C}(R)$ and $I \cup J'$, so less than $9N$ loops in $I \cup J'$ are are incident to only one connected component of $S \setminus (I \cup J')$, or are incident to a connected component of $S \setminus (I \cup J')$ whose corresponding sub-portalgon of R is not a tube. Among the other loops of J' less than N are based at a curved vertex. Every other loop is deleted in J.

This proves that after TUBING there are less than $19N$ inactive loops adjacent to $\mathcal{S}(R_A)$. Indeed all those loops belong to $I \cup J$. This also proves that after TUBING, in the interior of $\mathcal{S}(R_A)$, less than $10N$ vertices of $\mathcal{C}(R_A)$ are incident to a loop edge. Indeed every such vertex is incident to a loop in J, except in the particular case where before TUBING $\mathcal{S}(R_A)$ was a flat torus and contained exactly one vertex v incident to loop edges, in which case TUBING did not modify R_A and v remains in $\mathcal{C}(R_A)$.

Now after TUBING every vertex on the boundary of $\mathcal{S}(R_A)$ either lies on the boundary of S, and there are less than N such vertices as they all belong to the input portalgon, or is the base-vertex of an inactive loop, and there are less than $19N$ such vertices. So the boundary of $\mathcal{S}(R_A)$ has less than $20N$ vertices. In the interior of $\mathcal{S}(R_A)$ less than N vertices are curved, and less than $10N$ are incident to a loop edge. Altogether if $\mathcal{C}(R_A)$ has m_A vertices then the interior of $\mathcal{S}(R_A)$ has more than $m_A - 31N$ flat vertices not incident to any loop edge. Now assume $m_A > 744N$. Then $\mathcal{C}(R_A)$ has more than $m_A/12$ vertices of degree saller than or equal to six by Lemma 17, since the genus g_A and the number of boundary component b_A of $\mathcal{S}(R_A)$ satisfy $g_A \leq N$ and $b_A \leq 20N$, and since $m_A > 24 \times 21N$. So the interior of $\mathcal{S}(R_A)$ has more than $m_A/12 - 31N > m_A/24$ flat vertices of degree smaller than or equal to six not incident to any loop edge. That proves the claim, and the proposition. ◀

5.5 Proofs of Proposition 5 and Theorem 1

Proof of Proposition 5. Run the algorithm with $\kappa \geq 326$. We have two claims that imply the proposition. Our first claim is that the algorithm terminates in $O(n\log^2(n)\log^2(L/s))$ time, and that in the end R has $O(n\log(L/s))$ fragments. To prove this claim let n_A and L_A be respectively the maximum number of fragments, and the maximum fragment edge length reached by R_A during the execution of the algorithm.

The algorithm terminates in $O(n_A\log(n_A)\log(L_A/s)\log(L/s))$ time. Indeed SUBDIVISION and DELETION take $O(n_A)$ time. Also for every tube X simplified by TUBING, the systole of $\mathcal{S}(X)$ is greater than or equal to the systole of S, for otherwise one of the two loops of $\mathcal{C}(X)$ forming the boundary of $\mathcal{S}(X)$ would be contractible in S, and so would bound a topological disk in S, contradicting the Gauss-Bonnet formula. So TUBING takes $O(n_A\log(n_A)\log(L_A/s))$ time by Lemma 7.

In the end R has $O(n_A\log(L/s))$ fragments since each iteration removes $O(n_A)$ fragments from the active region R_A.

We have $n_A = O(n)$. Indeed $\mathcal{C}(R_A)$ has $O(n)$ vertices at any time by Proposition 15, since $\kappa \geq 326$. So $\mathcal{C}(R_A)$ has $O(n)$ edges by Lemma 18, and so R_A has $O(n)$ fragments.

We have $\log(L_A/s) = O(\log(n) + \log(L/s))$. Indeed at any time every edge e of $\mathcal{C}(R_A)$ longer than L must satisfy $c_S(e) < 60\kappa\log(L/s)$ by Proposition 14. Then $\ell(e)/s \leq 36000\kappa\log(L/s)n\lceil L/s\rceil^2$ by Proposition 8.

That proves the first claim. Our second claim is that in the end $h_S(e) = O(\log(n)\log^2(L/s))$ holds on every edge of e of $\mathcal{C}(R)$. Indeed if e is in $\mathcal{C}(R_A)$ then $c_S(e) \leq 60\kappa\log(L/s)$, for otherwise Proposition 14 would imply $\ell(e) < 2s$, implying that no loop encloses e in S, a contradiction. So $h_S(e) = O(\log(L/s)(\log(n) + \log(L/s)))$ by Proposition 8. Every other edge of $\mathcal{C}(R)$ belongs to the carrier of an inactive biface B. Every edge e of $\mathcal{C}(B)$ forming the boundary of $\mathcal{S}(B)$ is either a boundary component of S or an inactive loop, so $c_S(e) \leq 2$ by Lemma 13, so $h_S(e) = O(\log(n) + \log(L/s))$ by Proposition 8. Every edge f of $\mathcal{C}(B)$ in the interior of $\mathcal{S}(B)$ then satisfies $h_S(f) = O(\log(n) + \log(L/s))$ by Lemma 6. That proves the second claim, and the proposition. ◀

We just proved Proposition 5. In Appendix F we extend Proposition 5 to surfaces having positively curved points, essentially by cutting out caps around those points:

▶ **Proposition 19.** *Let P be a triangular portalgon, with n fragments, of global aspect ratio r. One can compute in $O(n\log^2(n)\log^2(r))$ time a triangular portalgon of $\mathcal{S}(P)$ that has $O(n\log(r))$ fragments, and happiness $O(n\log(n)\log^2(r))$.*

Our last technical result is independent, and proved in Appendix G as it is similar to previous work on polyhedral meshes:

▶ **Proposition 20.** *Let P be a triangular portalgon, with n fragments, of happiness h, whose surface $\mathcal{S}(P)$ is closed. One can compute the canonical portalgon of $\mathcal{S}(P)$ in $O^*(n^2h)$ time.*

Roughly, the proof of Proposition 20 goes by deducing the canonical portalgon from the Voronoi diagram of its vertices, which we compute by adapting the shortest path algorithm of [12, Section 3]. Theorem 1 is immediate from Proposition 19 and Proposition 20:

Proof of Theorem 1. Proposition 19 computes in $O^*(n \log^2 r)$ time a triangular portalgon P' of $\mathcal{S}(P)$ that has $O(n \log r)$ fragments, and happiness $O^*(n \log^2 r)$. Proposition 20 then computes from P' the canonical portalgon of $\mathcal{S}(P)$ in $O^*(n^3 \log^4 r)$ time. ◀

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A Appendix of Section 1

Proof of Lemma 2. Clearly $r' \leq r$. For the other inequality let e be a longest fragment edge of P, and let F be fragment whose smallest height d is minimum. Then $r = \ell(e)/d$. Since $\mathcal{S}(P)$ is connected there is a sequence of fragment edges $e_0, \ldots, e_{2k}$ for some $0 \leq k < p$ such that e_0 belongs to F, $e_{2k} = e$, for every $0 \leq i < k$ the edge e_{2i} is matched with the edge e_{2i+1}, and the edges e_{2i+1} and e_{2i+2} belong to the same fragment. Then $\ell(e_{2i+2}) \leq r' \cdot \ell(e_{2i+1})$ and $\ell(e_{2i+1}) = \ell(e_{2i})$. So $\ell(e)/d \leq \ell(e_0)(r')^{p-1}/d \leq (r')^p$. ◀

Proof of Corollary 3. Theorem 1 computes the canonical portalgon P' of $\mathcal{S}(P)$ in $O^*(n^3 \log^4 r)$ time. Cut the fragments of P' into triangles along arcs to get a triangular portalgon P''. By definition of the canonical portalgon the graph $\mathcal{C}(P'')$ is a Delaunay triangulation of $\mathcal{S}(P)$ whose vertex set is either a single point or the set of curved points of $\mathcal{S}(P)$. So P'' has $O(n)$ fragments, and P'' has bounded happiness by [12, Section 4]. ◀

Proof of Corollary 4. Theorem 1 computes the respective canonical portalgons $\mathcal{P}$ and $\mathcal{P}'$ of $\mathcal{S}(P)$ and $\mathcal{S}(P')$ in $O^*(n^3 \log^4 r)$ time. The two canonical portalgons have $O(n)$ fragment edges, so we can determine if they are equal in $O(n^2)$ time as follows. Fix a fragment edge e of $\mathcal{P}$. For every fragment edge e' of $\mathcal{P}'$ determine in $O(n)$ time if there is a one-to-one correspondence φ from the fragment edges of $\mathcal{P}$ to the fragment edges of $\mathcal{P}'$ that maps e to e', that maps the boundary closed walks of the fragments of $\mathcal{P}$ to the boundary closed walks of the fragments of $\mathcal{P}'$, and the matching of $\mathcal{P}$ to the matching of $\mathcal{P}'$, or correctly assert that there is none. If φ exists (then φ is unique since $\mathcal{S}(\mathcal{P})$ and $\mathcal{S}(\mathcal{P}')$ are connected) construct φ in $O(n)$ time. Then determine in $O(n)$ time if for every fragment F of $\mathcal{P}$ there is an orientation-preserving isometry $\tau_F : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $\varphi(e) = \tau_F(e)$ on every edge e of F. In which case return correctly that $\mathcal{P}$ and $\mathcal{P}'$ are equal. In the end, if every directed edge e' of $\mathcal{P}'$ has been looped upon, and if no equality has been found, return correctly that $\mathcal{P}$ and $\mathcal{P}'$ are distinct. ◀

B Appendix of Section 2: proof of Lemma 6

Proof of Lemma 6. Let f be a shortest interior edge of B. Let $g \neq f$ be the other interior edge of B. Let p be a shortest path in $\mathcal{S}(B)$. The relative interior $\mathring{p}$ of p cannot intersect the relative interior of f twice for those intersections would be crossing and p and f are both shortest paths since B is good. So $\mathring{p}$ intersects f less than four times. Then $\mathring{p}$ cannot intersect the relative interior of g five times, for those intersections would be crossings, and $\mathring{p}$ would intersect f in-between any two consecutive crossings with the relative interior of g. Altogether p intersects f and g less than seven times each. ◀

C Appendix of Section 3: proof of Lemma 7

In this section we prove Lemma 7. We need a few lemmas. Our starting point is a corollary of [12, Theorem 21]:

▶ **Lemma 21** (Corollary of [12, Theorem 21])**.** *Let B be a biface of happiness h. One can compute in $O(\log h)$ time a good biface of $\mathcal{S}(B)$.*

Proof. Apply [12, Theorem 21] to compute in $O(\log h)$ time a portalgon P of $\mathcal{S}(B)$ with $O(1)$ fragment edges, and whose happiness is smaller than or equal to 5. Also maintain which fragment vertices of P correspond to the two vertices b_0 and b_1 of $\mathcal{C}(B)$.

We now describe how to compute, in constant time, from P, a good biface of $\mathcal{S}(P)$. First compute, in constant time, by brute force, a shortest path q between b_0 and b_1: represent q by its pre-image in the fragments of P. Cut the fragments of P along the pre-image of q: every time a fragment is cut in two along a segment a, the two fragment edges issued of a are not matched in the resulting portalgon (the goal is to cut the surface of P, not just changing P). Consider the resulting portalgon D. Then $\mathcal{S}(D)$ is homeomorphic to a closed disk. The two endpoints b_0 and b_1 of q become a set V of four vertices of $\mathcal{C}(D)$ that lie on the boundary of $\mathcal{S}(D)$. Every curved point of $\mathcal{S}(D)$ lies on the boundary of $\mathcal{S}(D)$ and belongs to V. Now replace D by a triangular portalgon D' of $\mathcal{S}(D)$, such that V is the vertex-set of $\mathcal{C}(D')$, in constant time. This can be done for example by iteratively inserting arcs in the faces of $\mathcal{C}(D)$ to make $\mathcal{C}(D)$ a triangulation, and by deleting a vertex v of $\mathcal{C}(D)$ and its incident edges (as in Section 3). When v lies on the boundary of $\mathcal{S}(D)$, only the edges relatively included in the interior of $\mathcal{S}(D)$ are deleted. In the end, identify back the occurences of q on the boundary of $\mathcal{S}(D')$, by matching the two corresponding fragment edges in D', thereby obtaining a biface B' of $\mathcal{S}(B)$ such that q is an interior edge of B'. Change the other interior edge $f \neq q$ of B' if this is possible (equivalently, if the quadrilateron $\mathcal{S}(B') \setminus q$ admits two diagonals instead of just one), and if this shortens f. Then B' is good. ◀

Consider $k \geq 1$ bifaces $B_1, \ldots, B_k$. For every $1 \leq i \leq k$ let e_i and f_i be the two boundary edges of B_i. If $i < k$, assume $\ell(e_i) = \ell(f_{i+1})$, and match e_i with f_{i+1}. The resulting triangular portalgon is a ***concatenation*** of the bifaces $B_1, \ldots, B_k$.

▶ **Lemma 22.** *Let P be the concatenation of two good bifaces. One can compute in constant time a good biface of $\mathcal{S}(P)$.*

Proof. Consider a shortest path p in $\mathcal{S}(P)$, and the loop edge e in the interior of $\mathcal{C}(P)$ in-between the surfaces of the two bifaces. We claim that the relative interior of p does not cross the relative interior of e more than two times. By contradiction assume that p crosses the relative interior of e three times. There is a connected component S_0 of $\mathcal{S}(P) \setminus e$ whose angle at the base vertex of e is greater than or equal to π. Some portion p' of p enters S_0 and then leaves S_0 by two of the three crossings between p and e. One of the two connected components of $S_0 \setminus p'$, say S_1, is homeomorphic to an open disk. Then S_1 has at most three angles distinct from π, so they are smaller than π by the Gauss-Bonnet formula, and one of them is the incidence of S_0 and the base vertex of e. This is a contradiction. That proves the claim.

Using the claim immediately the intersection of p and e has $O(1)$ connected components, so p writes as a concatenation of $k = O(1)$ paths $p_1, \ldots, p_k$ such that for every $1 \leq i \leq k$ the path p_i is either included in e or relatively disjoint from e. Every edge $f \neq e$ of $\mathcal{C}(P)$ intersects p_i less than 7 times: if f is included in the boundary of $\mathcal{S}(P)$ then f intersects p_i at most once, otherwise Lemma 6 applies. So f intersects p less than $O(1)$ times. We proved that the segment-happiness of P, and thus the happiness of P since P is triangular, is $O(1)$. So we can compute a good biface of $\mathcal{S}(P)$ in constant time, exactly as in the proof of Lemma 21. ◀

The following consequence of the Euler formula is similar to Lemma 17:

▶ **Lemma 23.** *Let Y be a triangular portalgon whose surface $\mathcal{S}(Y)$ is homeomorphic to an annulus, such that $\mathcal{C}(Y)$ has one vertex on each boundary component of $\mathcal{S}(Y)$. At least half* *of the vertices of $\mathcal{C}(Y)$ that lie in the interior of $\mathcal{S}(Y)$ and are not incident to any loop edge have degree smaller than or equal to six.*

Proof. We may assume without loss of generality that no vertex of $\mathcal{C}(Y)$ in the interior of $\mathcal{S}(Y)$ is incident to a loop edge, by cutting $\mathcal{S}(Y)$ open at an interior loop edge (un-matching the two corresponding fragment edges of Y) and recursing on the resulting two triangular portalgons otherwise. Euler formula gives $m - m_1 + m_2 = 0$, where m, m_1, and m_2 count respectively the vertices, edges, and faces of $\mathcal{C}(Y)$. Double counting gives $3m_2 = 2m_1 - 2$ and $\sum_v \deg v = 2m_1$, where the sum is over the vertices v of $\mathcal{C}(Y)$. Then $\sum_v (6 - \deg v) = 4$. The two vertices of $\mathcal{C}(Y)$ on the boundary of $\mathcal{S}(Y)$ have degree greater than or equal to four. So in the interior of $\mathcal{S}(Y)$ every vertex of degree greater than six must be compensated by a vertex of degree smaller than six. ◀

Now we start proving Lemma 7. In particular we fix a tube X with n fragments, of maximum fragment edge length L.

▶ **Lemma 24.** *One can compute in $O(n \log n)$ time a concatenation of less than $3n$ bifaces, whose surface is isometric to $\mathcal{S}(X)$, whose edges are all shorter than $(3n)^c L$ with $c = \log_{14/13} 3$.*

Proof. Let us first describe the algorithm before analysing it. As long as there are vertices of $\mathcal{C}(X)$ in the interior of $\mathcal{S}(X)$ that are not incident to any loop edge and have degree smaller than or equal to six, we consider a maximal independent set V of such vertices, and we do the following. First we delete all the vertices in V along with their incident edges. Then we insert arbitrary arcs in the faces of $\mathcal{C}(X)$ to make $\mathcal{C}(X)$ a triangulation again.

The algorithm terminates since the number of vertices of $\mathcal{C}(X)$ decreases at each iteration. In the end every vertex in the interior of $\mathcal{S}(X)$ is incident to a loop edge by Lemma 23, so X is a concatenation of less than m bifaces, where $m \leq 3n$ is the initial number of vertices of X. Each iteration can be performed in $O(n)$ time by maintaining a bucket with the vertices of degree smaller than or equal to six. And we claim that there less than $\log_{14/13} m$ iterations. Before proving the claim, observe that it implies the lemma. Indeed the algorithm then terminates in $O(n \log n)$ time. Also no edge can get longer than $3^{\log_{14/13} m} L = m^c L$ since the maximum edge length of $\mathcal{C}(X)$ cannot be multiplied by more than 3 at each iteration.

Let us now prove the claim. Consider the number m' of vertices of $\mathcal{C}(X)$ not incident to any loop edge that lie in the interior of $\mathcal{S}(X)$. By Lemma 23, if $m' > 0$ before an iteration of the algorithm, then at least $m'/2$ such vertices have degree smaller than or equal to six. So V contains at least $m'/14$ vertices, which are deleted. Every non-deleted vertex that was incident to a loop edge before the iteration remains incident to a loop edge after the iteration. We proved that m' is divided by at least $14/13$ during the iteration, which proves the claim. ◀

Proof of Lemma 7. Apply Lemma 24, and replace X in $O(n \log n)$ time by a concatenation of less than $3n$ bifaces whose edges are smaller than $(3n)^c L$ for some constant $c > 0$. Each biface B has segment-happiness $O((3n)^c L/s)$; Indeed the systole of $\mathcal{S}(B)$ is greater than or equal to the systole of X, so every segment e in $\mathcal{S}(B)$ satisfies $h_{\mathcal{S}(B)}(e) = O(\ell(e)/s)$. Replace B by a good biface of $\mathcal{S}(B)$ in $O(\log(n) + \log(L/s))$ time with Lemma 21. Doing so for all bifaces takes $O(n(\log(n) + \log(L/s)))$ time in total. In the end apply Lemma 22 repeatedly to compute, from those $O(n)$ good bifaces, a single good biface of $\mathcal{S}(X)$, in $O(n)$ total time. ◀

D Appendix of Section 4

D.1 Proof of Proposition 8

In this section we prove Proposition 8. First we need a few lemmas.

▶ **Lemma 25.** *Let $t > 1$. Assume that there is a shortest path whose relative interior crosses the relative interior of e twice in the same direction, at points x and y. If the sub-segment of e between x and y is shorter than $\langle x \rangle_e / 2t$ then $c_S(e) \geq t$.*

Proof. Consider the portion p of the shortest path that starts just before its crossing at x, and ends just before its crossing at y. Consider also a geodesic path q that runs parallel to the sub-segment of e from y to x, and let γ be the concatenation of p and q. Then γ is non-contractible (since the interior of S has no positively curved point), and shorter than $\langle x \rangle_e / t$. Base γ at x, and let γ' be the geodesic loop homotopic to γ (where the basepoint at x is fixed in the homotopy). Then γ' is not longer than γ. In particular γ' is in general position with e.

We shall now prove that γ' meets x on both sides of e. To do so, orient e so that γ crosses e from right to left. Consider the universal covering space $\widetilde{S}$ of S, and a lift $\widetilde{e}$ of e in $\widetilde{S}$. Let $\widetilde{x}$ be the lift of x in $\widetilde{e}$. Two lifts of γ' are incident to $\widetilde{x}$: one starts at $\widetilde{x}$, the other ends at $\widetilde{x}$. Let $\widetilde{\gamma}'$ be the lift of γ' that starts at $\widetilde{x}$. We claim that $\widetilde{\gamma}'$ leaves $\widetilde{x}$ on the left side of $\widetilde{e}$. Let us prove the claim. Since the interior of $\widetilde{S}$ contains no positively curved point, there is a geodesic $\widetilde{L}$, containing $\widetilde{e}$, such that on both ends $\widetilde{L}$ is either infinite or reaches the boundary of $\widetilde{S}$. Then $\widetilde{L}$ separates $\widetilde{S}$ in two connected components. Consider the endpoint $\widetilde{a}$ of $\widetilde{\gamma}'$. Consider also the lift $\widetilde{p}$ of p that starts at $\widetilde{x}$, and the lift $\widetilde{q}$ of q that starts at the endpoint of $\widetilde{p}$. Then $\widetilde{q}$ ends at $\widetilde{a}$. Also $\widetilde{p}$ is disjoint from $\widetilde{L}$ except for its start point at $\widetilde{x}$ (recall that the interior of S has no positively curved point). Moreover $\widetilde{q}$ is disjoint from $\widetilde{L}$. For otherwise $\widetilde{q}$ would intersect $\widetilde{L}$ at a point $\widetilde{b}$ whose distance to $\widetilde{x}$ would be smaller than $\langle x \rangle_e / t$. But then the sub-segment of $\widetilde{L}$ between $\widetilde{b}$ and $\widetilde{x}$ would be no longer, and so would be included in $\widetilde{e}$. In particular $\widetilde{q}$ and $\widetilde{e}$ would intersect, a contradiction. We proved that $\widetilde{a}$ lies strictly to the left of $\widetilde{L}$. Then $\widetilde{\gamma}'$ leaves $\widetilde{x}$ on the left of $\widetilde{L}$, proving the claim. Similar arguments show that the lift of γ' ending at $\widetilde{x}$ meets $\widetilde{x}$ on the right of $\widetilde{L}$. That proves that γ' meets x on both sides of e. ◀

Recall that in this paper $\log(\cdot)$ denotes $\log_2(\lceil \cdot \rceil) + 1$.

▶ **Lemma 26.** *Holds $h_S(e) \leq 24 c_S(e) \log(\ell(e)/s)$.*

Proof. Let $t > 1$. Assume that in S there is a shortest path p that intersects e more than $24t \log(\ell(e)/s)$ times. Cut e at its middle point. One of the two resulting sub-segments of e, say f, intersects p more than $12t \log(\ell(e)/s)$ times. Partition f into sub-segments $f_0, f_1, \ldots, f_n$ for some $n \leq \log(\ell(e)/s)$, where the sub-segment f_0 contains the points $x \in f$ such that $\langle x \rangle_e \leq s/4$, and where for every $1 \leq i \leq n$ the sub-segment f_i contains the points $x \in f$ such that $2^{i-3}s \leq \langle x \rangle_e \leq 2^{i-2}s$. There is $0 \leq i \leq n$ such that p intersects f_i more than $6t$ times, since $6tn \leq 12t \log(\ell(e)/s)$. Then the relative interior of p crosses f_i twice (at least) in the same direction at points x and y, such that the sub-segment of f_i between x and y is shorter than $2^{i-4}s/t$, since $\ell(f_i) \leq 2^{i-3}s$. Also $i \geq 1$ as no shortest path crosses f_0 twice, since $\ell(f_0) < s/2$ (recall that the interior of S has no positively curved point). In particular $\langle x \rangle_e \geq 2^{i-3}s$. Then $c_S(e) \geq t$ by Lemma 25. ◀

▶ **Lemma 27.** *Holds $\ell(e) \leq 600 c_S(e) n \lceil L/s \rceil L$.*

Proof. Let $t > 1$. Assume $\ell(e) \geq 600tn\lceil L/s \rceil L$. We will prove that $c_S(e) \geq t$. This will prove the proposition since $c_S(e) \geq 1$. To do so let $d = 120n\lceil L/s \rceil L$. Cut e into three segments, a middle segment e_0 of length d, and two peripheral segments each longer than $2td$. We claim that there is in S a shortest path crossing the relative interior of e_0 twice in the same direction. This claim implies $c_S(e) \geq t$ by Lemma 25, which proves the proposition.

Let us prove the claim. Consider a triangular portalgon P of S with n fragments and of maximum fragment edge length L. Cut each edge of $\mathcal{C}(P)$ into $2\lceil L/s \rceil$ equal-length segments, that we shall call *doors*. Each door is smaller than or equal to half the systole of S so it is a shortest path. There are at most $6n\lceil L/s \rceil$ doors since $\mathcal{C}(P)$ has at most $3n$ edges. Each sub-segment e_1 of length $4L$ of e_0 contains in its relative interior three points x_0, x_1, x_2 in this order such that $x_0 \notin p$, $x_1 \in p$, and $x_2 \notin p$ for some door p. The relative interior of e_0 intersects at least $30n\lceil L/s \rceil$ times doors this way, so there is a door p intersected at least 5 times by the relative interior of e_0. Then each intersection is a single point (p and e_0 do not overlap). Two of those intersection points may be end-points of p, but otherwise the relative interior of p crosses the relative interior of e_0 at least three times. So p crosses e_0 twice in the same direction, which proves the claim, and the proposition. ◀

Proof of Proposition 8. We have $h_S(e) \leq 24c_S(e)\log(\ell(e)/s)$ by Lemma 26 and $\ell(e)/s \leq 600c_S(e)n\lceil L/s \rceil^2$ by Lemma 27. Then $\log(\ell(e)/S) < 10(\log(c_S(e)) + \log(n) + 2\log(L/s))$. ◀

D.2 Proof of Lemma 9

Proof of Lemma 9. Let $t > 1$. Assume that there is a loop γ, based at a point x, that encloses f by a factor of t. Then γ encloses e by a factor of t since $\langle x \rangle_f \leq \langle x \rangle_e$. ◀

D.3 Proof of Proposition 10

In this section we prove Proposition 10. First we need a lemma:

▶ **Lemma 28.** *In S, let e be and f be two relatively disjoint segments, and let γ be a geodesic loop. Assume that γ encloses e by a factor of $t > 2$, and that γ intersects f at a point y such that $\langle y \rangle_f > \ell(\gamma)$. Rebase γ at y, and let γ' be the geodesic loop homotopic to it. Then γ' meets y on both sides of f.*

Proof. We have $\ell(\gamma') \leq \ell(\gamma)$ so $\ell(\gamma') < \langle y \rangle_f$, and so γ' is in general position with f. We prove the lemma by contradiction, so assume that γ' meets y only on the right side of f, for some direction of f. In the universal covering space $\widetilde{S}$ of S, consider a lift $\widetilde{f}$ of f. Let $\widetilde{y}$ be the lift of y that belongs to $\widetilde{f}$. Since the interior of $\widetilde{S}$ contains no positively curved point, there is a geodesic $\widetilde{L}$, containing $\widetilde{f}$, such that on both ends $\widetilde{L}$ is either infinite or reaches the boundary of $\widetilde{S}$. Then $\widetilde{L}$ separates $\widetilde{S}$ in two connected components. The two lifts of γ' incident to $\widetilde{y}$ meet $\widetilde{y}$ on the right side of $\widetilde{f}$ by assumption, and they are otherwise disjoint from $\widetilde{L}$. In particular, their other endpoints lie on the right side of $\widetilde{L}$.

We have $\ell(\gamma) < \langle y \rangle_f$ so γ is in general position with f. Direct γ so that γ crosses f from right to left at y, and write γ as the concatenation of two paths γ_0 and γ_1 respectively before and after its crossing at y. There is a lift $\widetilde{\gamma}_1$ of γ_1 that leaves $\widetilde{y}$ on the left of $\widetilde{f}$. And $\widetilde{\gamma}_1$ is otherwise disjoint from $\widetilde{L}$, since the interior of $\widetilde{S}$ has no positively curved point. Thus the endpoint $\widetilde{x}$ of $\widetilde{\gamma}_1$ lies on the left of $\widetilde{L}$. There is a lift $\widetilde{\gamma}_0$ of γ_0 that starts at $\widetilde{x}$. And $\widetilde{\gamma}_0$ is otherwise disjoint from $\widetilde{\gamma}_1$ since γ meets x on both sides of e, and since the interior of $\widetilde{S}$ has no positively curved point. By the previous paragraph, the endpoint of $\widetilde{\gamma}_0$ lies on the right side of $\widetilde{L}$, so $\widetilde{\gamma}_0$ intersects $\widetilde{L}$. Cut $\widetilde{\gamma}_0$ at its first intersection point $\widetilde{z}$ with $\widetilde{L}$. Let $\widetilde{I}$ be the sub-segment of $\widetilde{L}$ between $\widetilde{y}$ and $\widetilde{z}$. The concatenation of the prefix of $\widetilde{\gamma}_0$ ending at $\widetilde{z}$, of $\widetilde{I}$, and of $\widetilde{\gamma}_1$ is a simple closed curve $\widetilde{C}$. At $\widetilde{x}$, there is a portion of $\widetilde{e}$ that enters the bounded side of $\widetilde{C}$, since γ meets x on both sides of e. This portion of $\widetilde{e}$ can be extended into a geodesic $\widetilde{p}$ that meets $\widetilde{C}$ at some point $\widetilde{v}$, since the interior of $\widetilde{S}$ has no positively curved point. Then $\widetilde{v}$ belongs to the relative interior of $\widetilde{I}$. We claim that $\widetilde{v}$ belongs to the relative interiors of both $\widetilde{e}$ and $\widetilde{f}$, which is a contradiction since e and f are relatively disjoint. To prove the claim, first observe that the distance between $\widetilde{y}$ and $\widetilde{z}$ in $\widetilde{S}$ is at most $\ell(\gamma)$, and this distance is equal to the length of $\widetilde{I}$, since the interior of $\widetilde{S}$ has no positively curved point. So the sub-segment of $\widetilde{I}$ between $\widetilde{y}$ and $\widetilde{v}$ is no longer than $\ell(\gamma) < \langle y\rangle_f$, and is thus included in the relative interior of $\widetilde{f}$. Also, the distance between $\widetilde{v}$ and $\widetilde{x}$ is smaller than or equal to $2\ell(\gamma) \leq 2\langle x\rangle_e/t < \langle x\rangle_e$, so $\widetilde{p}$ is included in the relative interior of $\widetilde{e}$. ◀

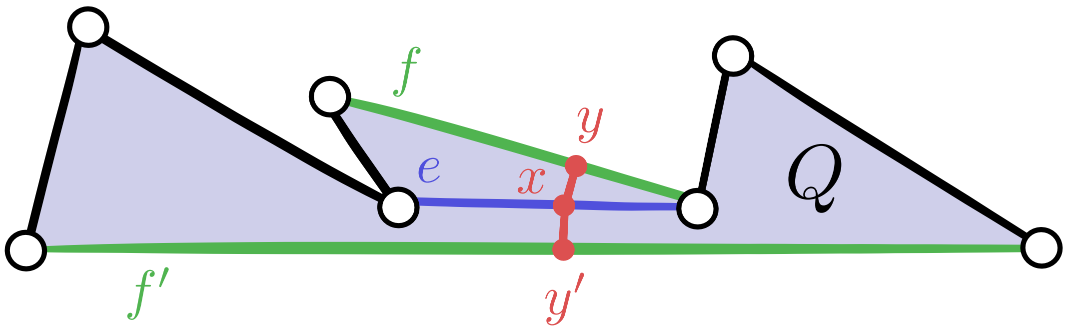

Figure 5 The setting of Lemma 29.

The proof of Proposition 10 also relies on the following construction. See Figure 5. In the Euclidean plane $\mathbb{R}^2$ let Q be a polygon with more than three vertices. Let e be a shortest arc of Q. Let f and f' be sides of Q separated by e along the boundary of Q. Let x be a point in the relative interior of e. Let y and y' be points that lie on respectively f and f' (possibly vertices of Q), and do not lie on e. Assume that the segments p and p' between x and respectively y and y' are relatively included in the interior of Q. Then:

▶ **Lemma 29.** *Let $t > 6$. If $\ell(p) \leq \langle x\rangle_e/t$ and $\ell(p') \leq \langle x\rangle_e/t$, then at least one of f and f', say f, is such that $\langle y\rangle_f \geq (1-4/t)\langle x\rangle_e$ and $\ell(f) \geq (1-4/t)\ell(e)$.*

Proof. Assume without loss of generality that e is horizontal, that f lies above e, and that x is the origin $(0,0) \in \mathbb{R}^2$. Then x cuts e into two segments e_0 and e_1, respectively the right and left one. Let v_0 and v_1 be respectively the right and left endpoints of e. Consider the following algorithm in three phases. In the first phase consider the point $z = x$ and move z along p. Doing so, consider the segments from z to v_0 and v_1. If moving z makes the relative interior of one of those two segments intersect ∂Q, then stop: this is a break condition. Also break if z reached y and y is a vertex of Q. Otherwise the algorithm enters its second phase. Then y cuts f in two segments f_0 and f_1, where f_0 is on the right of y as seen from the path p directed from x to y. In phase two move z along f_0 or f_1, choosing carefully which segment to move along so that the second coordinate of z does not increase. We assume without loss of generality that z moves along f_0, by flipping the figure horizontally otherwise. Move along f_0 by a distance of $(1-4/t)\ell(e_0)$, but break if z reaches the right end-vertex of f, or if the relative interior of the segment between z and v_0 intersects ∂Q. If the algorithm did not break, it enters its third and final phase. In this phase put z back on y, and move it along the other sub-segment of f, here f_1, by a distance of $(1-4/t)\ell(e_1)$, breaking if z reaches the left end-vertex of f, or if the relative interior of the segment between z and v_1 intersects ∂Q.

If the algorithm did not break then $\ell(f) \geq (1-4/t)\ell(e)$ and $\langle y\rangle_f \geq (1-4/t)\langle x\rangle_e$ and we are done. Otherwise, if the algorithm broke, consider the triangle Δ between v_0, v_1, and z. The break conditions ensure that the interior of Δ is included in the interior of Q, and that there is a vertex w of Q that lies on $\partial\Delta$ and not on e. We claim that the inner-angles of Δ at v_0 and v_1 are both strictly smaller than $\pi/4$. We prove this claim by considering the coordinates $(\alpha, \beta) \in \mathbb{R} \times [0, +\infty[$ of z, and the coordinates $(\ell(e_0), 0)$ and $(-\ell(e_1), 0)$ of v_0 and v_1 respectively, and by proving that the invariants $\ell(e_0)-\alpha > \beta$ and $\alpha+\ell(e_1) > \beta$ hold at any time during the algorithm. Let $m = \min(\ell(e_0), \ell(e_1)) = \langle x\rangle_e$. In the first phase $|\alpha| \leq m/t$ and $0 \leq \beta \leq m/t$, so the invariants hold since $t > 2$. In the second phase β does not increase and α does not decrease. Moreover α does not increase by more than $\ell(e_0)(1-4/t)$ so the invariants hold. If the second phase ends without breaking then the absolute slope λ of the line supporting f is smaller than or equal to $1/(t-5)$. Indeed during the second phase β decreased by at most m/t while z moved a distance $\ell(e_0)(1-4/t)$, so α increased by at least $\ell(e_0)(1-4/t) - m/t$, and so $1/\lambda \geq \ell(e_0)(1-4/t)t/m - 1 \geq t-5$. In the third phase $\alpha \geq -m/t - \ell(e_1)(1-4/t)$ and $\beta \leq m/t + \lambda\ell(e_1)(1-4/t)$ so $\alpha + \ell(e_1) \geq 3\ell(e_1)/t > \beta$ since $t > 6$. Also β increases less than α decreases since $\lambda < 1/2$, so $\ell(e_0) - \alpha > \beta$ remains true. That proves the claim.

Applying the algorithm to p' and f' on the other side of e, either the algorithm does not break in which case $\ell(f') \geq (1-4/t)\ell(e)$, $\langle y'\rangle_{f'} \geq (1-4/t)\langle x\rangle_e$, and we are done. Or the algorithm breaks and we get similarly a triangle Δ' and a vertex w' of P. The inner angles of Δ' at v_0 and v_1 are also both strictly smaller than $\pi/4$, so the segment between w and w' is relatively included in the interior of the quadrilateron formed by Δ and Δ', and is strictly shorter than e. This segment is an arc of Q shorter than e, a contradiction. ◀

Proof of Proposition 10. Let $t > 6$. Assume that there is a geodesic loop γ that encloses e by a factor of t. Let x be the basepoint of γ. In the Euclidean plane, consider the fragment Q corresponding to F. Let $\widehat{e}$ and $\widehat{x}$ be the pre-images of e and x in Q. Consider the prefix and the suffix of γ that leave x on both sides of e to meet ∂F, and their pre-image paths in Q that meet two boundary edges $\widehat{f}$ and $\widehat{f}'$ of Q, at respective points $\widehat{y}$ and $\widehat{y}'$. By Lemma 29, one of those two points, say $\widehat{y}$ without loss of generality, is such that $\langle\widehat{y}\rangle_{\widehat{f}} \geq (1-4/t)\langle\widehat{x}\rangle_{\widehat{e}}$ and $\ell(\widehat{f}) \geq (1-4/t)\ell(\widehat{e})$. Also $\widehat{f}$ projects to a boundary edge f of F, and $\widehat{y}$ projects to a point y in the relative interior of f. Now rebase γ at y, and consider the geodesic loop γ' homotopic to it (where the basepoint at y is fixed by the homotopy). Then $\ell(\gamma') \leq \ell(\gamma) = \langle x\rangle_e/t < \langle y\rangle_f/(t-4)$. In particular $\ell(\gamma') < \langle y\rangle_f$ since $t > 5$. And γ' meets y on both sides of f by Lemma 28, since $t > 2$. ◀

D.4 Proof of Proposition 11

In this section we prove Proposition 11. First we need a lemma:

▶ **Lemma 30.** *Let B be a good biface. Let f be a longest boundary edge of $\mathcal{C}(B)$, and let F be the face of $\mathcal{C}(B)$ adjacent to f. Each corner of F incident to f has angle smaller than or equal to $\pi/2$.*

Proof. Let e be a shortest interior edge of $\mathcal{C}(B)$, and let $g \neq e$ be the other interior edge of $\mathcal{C}(B)$. Then e, g, and f are the sides of F. The angle at the corner of F between f and g is smaller than $\pi/2$ since $\ell(e) \leq \ell(g)$. Now consider the corner c between f and e. Cut $\mathcal{S}(B)$ open along e and consider the resulting quadrilateron Q in the plane. The edge f of $\mathcal{C}(B)$ corresponds to a side $\widehat{f}$ of Q, the edge e corresponds to two opposite sides $\widehat{e}$ and $\widehat{e}'$, and the edge g corresponds to an arc $\widehat{g}$ of Q. Also the other boundary edge $f' \neq f$ of

$\mathcal{C}(B)$ corresponds to the side $\widehat{f}'$ of Q opposite to $\widehat{f}$. And the corner c corresponds to the corner $\widehat{c}$ of Q between $\widehat{e}$ and $\widehat{f}$. Let $\widehat{d}$ be the corner of Q opposite to $\widehat{c}$, between $\widehat{e}'$ and $\widehat{f}'$. Assume by contradiction that the angle at $\widehat{c}$ is greater than $\pi/2$. We have $\ell(\widehat{e}) = \ell(\widehat{e}')$ and $\ell(\widehat{f}) \geq \ell(\widehat{f}')$ so the angle at $\widehat{d}$ is greater than or equal to the angle at $\widehat{c}$, and in particular is also greater than $\pi/2$. The two other angles of Q are smaller than π, so Q is convex and admits a diagonal $p \neq \widehat{g}$. Consider the unique circle C that admits $\widehat{g}$ as a diameter. Then the two endpoints of p lie in the interior of C. So p is shorter than $\widehat{g}$. This contradicts the assumption that B is good. ◀

Proof of Proposition 11. Let g and g' be respectively a shortest interior edge and a longest boundary edge of B. Then $\ell(g) \leq \ell(g')$ since B is thick. We claim that if $c_S(g) > 2$, then $c_S(g') \geq c_S(g) - 1$. To prove the claim let $t > 2$ and assume that there is a loop γ that encloses g by a factor of t in S. Let x be the basepoint of γ. Let F be the face of $\mathcal{C}(B)$ adjacent to g'. Around x there is a portion of γ that enters F. This portion of γ must leave F by a point y of g' since the angle of F between g and g' is smaller than or equal to $\pi/2$ by Lemma 30, since $\ell(g) \leq \ell(g')$, and since $\ell(\gamma) = \langle x\rangle_g/t < \langle x\rangle_g/\sqrt{2}$. Then $\langle y\rangle_{g'} \geq \langle x\rangle_g - \ell(\gamma) = (1 - 1/t)\langle x\rangle_g$ by triangular inequality and since $\ell(g) \leq \ell(g')$. Rebase γ at y, and consider the geodesic loop γ' homotopic to it (where the homotopy fixes the basepoint at y). Then $\ell(\gamma') \leq \ell(\gamma) = \langle x\rangle_g/t \leq \langle y\rangle_{g'}/(t-1)$. And γ' encloses g' by Lemma 28, since $t > 2$. That proves the claim.

If $e = g$ we are done by our claim, so assume that e is a longest interior edge of $\mathcal{C}(B)$. Deleting e merges the two faces of $\mathcal{C}(B)$ into a single face F' of which e is a shortest arc, since B is good. So Proposition 10 applies since $c_S(e) > 6$: there is a boundary edge f of F' such that $c_S(f) \geq c_S(e) - 4$ and $\ell(f) \geq (1 - 4/c_S(e))\ell(e)$. If f is a boundary edge of $\mathcal{C}(B)$ we are done. Otherwise $f = g$ so $\ell(g') \geq \ell(f) \geq (1-4/c_S(e))\ell(e)$ and $c_S(g') \geq c_S(f) - 1 \geq c_S(e) - 5$ by our claim since $c_S(e) > 6$. That proves the proposition. ◀

D.5 Proof of Proposition 12

In this section we prove Proposition 12. First we need two lemmas:

▶ **Lemma 31.** *Let B be a thin biface. Among the two interior edges of $\mathcal{C}(B)$ let e be a shortest one. Each one of the four corners between e and the boundary of $\mathcal{S}(B)$ has angle greater than $\pi/4$.*

Proof. Assume by contradiction that there is a corner c between e and a boundary edge f of $\mathcal{C}(B)$ whose angle is smaller than or equal to $\pi/4$. Cut $\mathcal{S}(B)$ open along e and embed the resulting quadrilateron Q in the plane, isometrically. The edge e corresponds to two opposite sides $\widehat{e}$ and $\widehat{e}'$ of Q. The edge f corresponds to one of the other two sides of Q, that we call $\widehat{f}$. The vertex v of the corner c corresponds to the two end-vertices of $\widehat{f}$: let $\widehat{v}$ be the one incident to $\widehat{e}$, and let $\widehat{v}'$ be the one incident to $\widehat{e}'$. Without loss of generality the corner c corresponds to the corner of Q at $\widehat{v}$, whose angle is thus smaller than or equal to $\pi/4$. Consider the orthogonal projection x of $\widehat{v}'$ on the line containing $\widehat{e}$. Then x belongs to $\widehat{e}$ since $\widehat{e}$ is longer than $\widehat{f}$, as B is thin. The segment p between x and $\widehat{v}'$ is shorter than the portion of $\widehat{e}$ between x and $\widehat{v}$. Also p is included in Q since $\widehat{e}$ and $\widehat{e}'$ are longer than $\widehat{f}$. Thus p projects to a path that shortcuts e, contradicting the fact that B is a good biface. ◀

▶ **Lemma 32.** *In $\mathcal{S}(B)$ every path p between the two boundary components of $\mathcal{S}(B)$ is such that $\ell(p) \geq \ell(e)/2$.*

Proof. Without loss of generality one of the two endpoints of p (at least) is a vertex v of $\mathcal{C}(B)$. Consider the other endpoint x of p, and the vertex $w \neq v$ of $\mathcal{C}(B)$. There is a path q from x to w in the boundary of $\mathcal{S}(B)$. Without loss of generality $\ell(q) \leq \ell(e)/2$ since B is thin. Also e is a shortest path since B is good. So $\ell(p) + \ell(q) \geq \ell(e)$. We proved $\ell(p) \geq \ell(e)/2$. ◀

Proof of Proposition 12. Let e be a shortest interior edge of $\mathcal{C}(B)$, and let f be a boundary edge of $\mathcal{C}(B)$. We have $\ell(e) \geq \ell(f)$ since B is thin. Assume by contradiction that there is in S a loop γ that encloses f by a factor of $t > 2$. Let x be the basepoint of γ. There is a portion of γ that leaves x and enters the interior of $\mathcal{S}(B)$. This portion of γ cannot leave $\mathcal{S}(B)$ via the other boundary edge of $\mathcal{S}(B)$, for otherwise $\ell(\gamma) \geq \ell(e)/2$ by Lemma 32, so $\ell(\gamma) > \langle x \rangle_f / t$, a contradiction. Then γ intersects e. And f and e have a corner whose angle is smaller than $\pi/4$ since $\ell(\gamma) < \langle x \rangle_f / 2$. This contradicts Lemma 31. ◀

E Appendix of Sections 5.3 and 5.4

E.1 End of proof of Proposition 14

End of proof of Proposition 14. All there remains to do is to prove the second and third claims. First we recall the second claim. Let R' result from applying DELETION to R, and assume that there is an edge e' in $\mathcal{C}(R'_A)$ such that $c_S(e') > 13$. Our second claim was that there is an edge e in $\mathcal{C}(R_A)$ such that $c_S(e) \geq c_S(e') - 12$ and $\ell(e) \geq (1 - 12/c_S(e'))\ell(e')$. Now we prove the second claim. Assume that e' does not belong to $\mathcal{C}(R_A)$, for otherwise we are done. Then e' was inserted in a face F by the routine, where F results from the DELETION of a vertex v and its incident edges. At most 3 arcs were inserted in F since the degree of v was smaller than or equal to six. By applying Proposition 10 at most three times, we get that there is a boundary edge e of F such that $c_S(e) \geq c_S(e') - 12$ and $\ell(e) \geq (1 - 12/c_S(e'))\ell(e')$. And e is an edge of $\mathcal{C}(R_A)$. That proves the second claim.

Now we recall the third claim. Let R' result from applying TUBING to R, and assume that there is an edge e' in $\mathcal{C}(R'_A)$ such that $c_S(e') > 6$. Our third claim was that there is an edge e in $\mathcal{C}(R_A)$ such that $c_S(e) \geq c_S(e') - 5$ and $\ell(e) \geq (1 - 4/c_S(e'))\ell(e')$. Now we prove the third claim. Assume that e' does not belong to $\mathcal{C}(R_A)$, for otherwise we are done. Then e' is an interior edge of a good biface B computed by the routine in step 3. And B is thick for otherwise B would have been removed from R_A by the routine, and marked as an inactive region. So by Proposition 11 there is a boundary edge e of B such that $c_S(e) \geq c_S(e') - 5$ and $\ell(e) \geq (1 - 4/c_S(e'))\ell(e')$. And e is an edge of $\mathcal{C}(R_A)$. That proves the third claim. ◀

E.2 Proof of Lemma 16

Proof of Lemma 16. Cut $\mathcal{S}(Y)$ along I, and consider the resulting surfaces, and their corresponding sub-portalgons of Y. Let Z contain those portalgons, and let $Z' \subseteq Z$ contain those that are not tubes. Without loss of generality $I \neq \emptyset$. Then every $Y_0 \in Z$ is such that $\partial\mathcal{S}(Y_0) \neq \emptyset$ since $\mathcal{S}(Y)$ is connected. Let $\chi(Y_0)$, $c(Y_0)$, and $d(Y_0)$ be respectively the Euler characteristic of $\mathcal{S}(Y_0)$, the number of curved points in the interior of $\mathcal{S}(Y_0)$, and the number of boundary components of $\mathcal{S}(Y)$ that belong to $\mathcal{S}(Y_0)$. Let $\lambda(Y_0) = 2c(Y_0) + 2d(Y_0) - \chi(Y_0)$. We claim that every $Y_0 \in Z$ satisfies $\lambda(Y_0) \geq 0$, and that if $Y_0 \in Z'$ then $\lambda(Y_0) > 0$. Indeed we have $\chi(Y_0) \leq 1$ since $\mathcal{S}(Y_0)$ is not homeomorphic to a sphere. So assuming $\lambda(Y_0) \leq 0$, we get $c(Y_0) = d(Y_0) = 0$. Then $\chi(Y_0) \neq 1$ for otherwise $\mathcal{S}(Y_0)$ would be homeomorphic to a disk, would have no curved point in its interior, and would be bounded by a single geodesic loop (issued of I), contradicting Gauss-Bonnet Formula. So $\chi(Y_0) = 0$. Then Y_0 is a tube since

$\mathcal{S}(Y_0)$ is not homeomorphic to a torus. That proves the claim. Now for every $Y_0 \in Z'$ let $b(Y_0)$ be the number of boundary components of $\mathcal{S}(Y_0)$. The claim implies $b(Y_0) \leq 2 - \chi(Y_0) \leq 2 + \lambda(Y_0) \leq 3\lambda(Y_0)$. So $\sum_{Y_0 \in Z'} b(Y_0) \leq 3\sum_{Y_0 \in Z'} \lambda(Y_0) \leq 3\sum_{Y_0 \in Z} \lambda(Y_0) \leq 9(g+b+c)$. Therefore at most $9(g+b+c)$ loops in I are incident to the surface of some $Y_0 \in Z'$. If every other loop in I is incident to the surfaces of two distinct $Y_0, Y_1 \in Z$ then we are done. Otherwise there is a loop $e \in I$ incident to the surface of only one $Y_0 \in Z$, and such that Y_0 is a tube ($Y_0 \notin Z'$). Then $\mathcal{S}(Y)$ is a flat torus and e is the only loop in I, so we are done. That proves the lemma. ◀

E.3 Proof of Lemma 17

Proof of Lemma 17. Let m_1 and m_2 count respectively the edges and the faces of the triangulation. Euler formula gives $6m - 6m_1 + 6m_2 = 12 - 12g - 6b$. Double counting gives $3m_2 \leq 2m_1 - b$ and $2m_1 = \sum_v \deg v$, where the sum is over the vertices, and where $\deg v$ denotes the degree of a vertex v. Then $\sum_v 6 - \deg v = 6m - 2m_1 \geq 6m - 6m_1 + 6m_2 + 2b \geq 12 - 12g - 4b > -m/2$. Let a and b count the number of vertices whose degree is respectively smaller than or equal to 6, and greater than six. Then $b < 5a + m/2$. Assuming $a < m/12$, we get $b < 11m/12$, and so $a + b < m$. This is a contradiction. ◀

E.4 Proof of Lemma 18

Proof of Lemma 18. Let m_1 and m_2 count respectively the edges and the faces of the triangulation, and let b count its boundary components. Double counting gives $3m_2 \leq 2m_1$. Euler formula gives $m_1 - m_2 = m + 2g + b - 2$. And we have $b \leq m$. Therefore $m_1 \leq 3m_1 - 3m_2 < 6(m+g)$. ◀

F Proof of Proposition 19

In this section we deduce Proposition 19 from Proposition 5, casting off the requirement of non-positive curvature of Proposition 5. Essentially, we cut out caps around the positively curved vertices, apply Proposition 5 to the truncated portalgon, and we put the caps back.

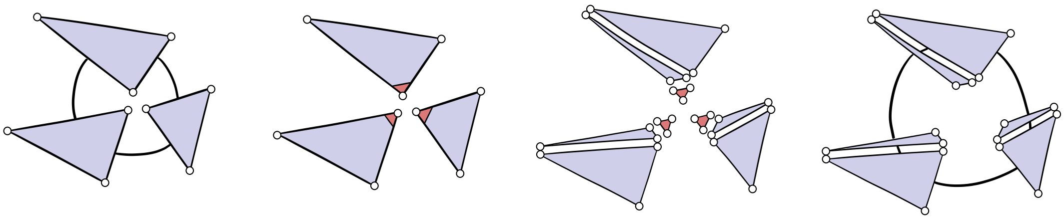

Figure 6 Cutting out a cap in the proof of Proposition 19.

Proof of Proposition 19. Let $S := \mathcal{S}(P)$ be the surface of P. Let d be the minimum height of the fragments of P. Given a vertex v of $\mathcal{C}(P)$ in the interior of S, we define a region around v in S, as follows. On every directed edge e of $\mathcal{C}(P)$ whose tail is v, place a point at distance $d/6$ from the tail of e along e. Link those $k \geq 1$ points in order around v, using geodesic segments within the faces of $\mathcal{C}(P)$ incident to v. In each corner of $\mathcal{C}(P)$ incident to v there is a newly created triangle incident to v. Those k triangles define a region around v, that we call *cap* C of v. Importantly, every point in the cap of v is at distance smaller than or equal to $d/6$ from v in S. Also every segment p tracing the boundary of C satisfies

$\ell(p) \geq d/6r$. To see that consider the face F of $\mathcal{C}(P)$ containing p, and the two sides e_0 and e_1 of F incident to v. For each i consider the point on e_i at distance $m := \min(\ell(e_0), \ell(e_1))$ from v along e_i. Join those two points by a geodesic segment q in F. Then q is at least as long as the minimum height of the fragment corresponding to F, and $\ell(q)/\ell(p) = 6m/d$ by Thales theorem. So $\ell(p) \geq d/6r$.

For the sake of analysis, given an arbitrary vertex v of $\mathcal{C}(P)$ (possibly on the boundary of S), we define an other kind of region around v. Link the middle points of the edges around v in order around v. The resulting triangles around v constitute the *protected region* of v. Importantly, every path smaller than $d/2$ starting from v must lie in the protected region of v. Indeed every geodesic path p smaller than d starting from v is relatively included in a single face or edge of $\mathcal{C}(P)$. Then every prefix of p smaller than $\ell(p)/2$ lies in the protected region of v.

First construct in $O(n)$ time a triangular portalgon P_0 of S, as follows. Consider every positively curved vertex v in the interior of $\mathcal{C}(P)$ (if any), and trace the boundary of the cap around v in the fragments of P. Then cut the fragments along the trace, as in Figure 6. Afterward some fragments of P_0 may not be triangles, so cut them along arcs. Now remove the fragments corresponding to the caps from P_0, and let P_1 be the resulting triangular portalgon. The interior of $\mathcal{S}(P_1)$ has no positively curved vertex. If moreover $\mathcal{S}(P_1)$ is simply connected then every edge of $\mathcal{C}(P_1)$ is the unique shortest path between its endpoints, so the segment-happiness of P_1 is 1.

Our first claim is that if $\mathcal{S}(P_1)$ is not simply connected, then the systole of $\mathcal{S}(P_1)$ is greater than or equal to $d/6r$. By contradiction assume that there is a non-contractible closed curve γ in $\mathcal{S}(P_1)$ smaller than $d/6r$. Without loss of generality γ intersects a vertex w of $\mathcal{C}(P_1)$. If w is a vertex of $\mathcal{C}(P)$, then γ lies in the protected region around w, and so γ is contractible in $\mathcal{S}(P_1)$, a contradiction. If w is a vertex on the boundary of some cap C removed, then γ lies in the protected region around the central vertex of C. In that case γ is at least as long as any edge of the boundary of C, so $\ell(\gamma) \geq d/6r$. That proves the first claim.

The number of fragments and the maximum fragment edge length of P_1 may be greater than those of P, but only by a constant factor. Using the first claim and Proposition 5, replace P_1 by a triangular portalgon of $\mathcal{S}(P_1)$ with $O(n \log(r))$ fragments, whose segment-happiness is $O(\log(n) \log^2(r))$, all in $O(n \log^2(n) \log^2(r))$ time. Place back the caps on $\mathcal{S}(P_1)$, and return the resulting triangular portalgon P'.

Our second claim is that the segment-happiness of P', and thus the happiness of P' since P' is triangular, is bounded by $O(n \log(n) \log^2(r))$. To prove the second claim, we call *cap path* any shortest path in S that lies in the closure of some cap. We call *rogue path* any shortest path in S whose relative interior is disjoint from the closures of the caps. Every rogue path intersects every edge of $\mathcal{C}(P')$ at most $O((\log n) \log^2 r)$ times, since the segment-happiness of P_1 is $O((\log n) \log^2 r)$. Also every cap path intersects every edge of $\mathcal{C}(P')$ at most once. Now consider a shortest path p in S. Then p uniquely writes as a sequence X of alternatively cap paths and rogue paths. Also, there cannot be two distinct cap paths q_0 and q_1 in X that both lie in the same cap C. For otherwise any point of q_0 would be at distance at most $d/3$ from any point of q_1. Also the subpath of p between q_0 and q_1 contains a rogue path, that must leave the protected region around the central vertex of C, and is thus longer than $d/2 - d/6 = d/3$. That contradicts the fact that p is a shortest path. We proved that there are at most $O(n)$ paths in X, each intersecting at most $O((\log n) \log^2 r)$ times any given edge of $\mathcal{C}(P')$. That proves the second claim, and the proposition. ◀

G Proof of Proposition 20

In this section we prove Proposition 20. We fix throughout a triangular portalgon P with n fragments, of happiness h, whose surface $S := \mathcal{S}(P)$ is closed. We let V contain the curved points of S (all of them are vertices of $\mathcal{C}(P)$) if there are any. Otherwise, if S is a flat torus, we let V contain a single arbitrary vertex of $\mathcal{C}(P)$.

The canonical portalgon of S is the one obtained by cutting S open along the Delaunay tessellation $\mathcal{D}$ of (S, V). We compute $\mathcal{D}$ from the Voronoi diagram $\mathcal{V}$ of (S, V). The duality between the two is classical in the plane [4]. To compute $\mathcal{V}$ we slightly extend the single-source shortest path algorithms of [12] to multiple-sources, adopting a strategy similar to that of [16] on polyhedral meshes.

G.1 Preliminaries on the Delaunay tessellation

We give the definition of Bobenko and Springborn [1] of the Delaunay tessellation of (S, V). An ***immersed empty disk*** is a pair (D, φ) where D is an open disk in the plane, and $\varphi : \overline{D} \to S$ is a map whose restriction to D is an isometric immersion such that $\varphi(D) \cap V = \emptyset$. (Note that φ is in general not injective.) Then:

▶ **Lemma 33** (Proposition 4 of [1]). *There is a unique tessellation $\mathcal{D}$ of S such that for every immersed disk (D, φ), if $\varphi^{-1}(V)$ is not empty, then the convex hull of $\varphi^{-1}(V)$ projects to either a vertex, an edge, or the closure of a face of $\mathcal{D}$, and such that every vertex, edge, and face of $\mathcal{D}$ can be obtained this way.*

The tessellation $\mathcal{D}$ given by Lemma 33 is the ***Delaunay tessellation*** of (S, V). In preparation for future work we also consider the following definition. For every point $x \in S$ there is an immersed empty disk (D, φ) such that φ maps the center of D to x, and such that $\varphi^{-1}(V) \neq \emptyset$. And (D, φ) is unique to x in the sense that if (D', φ') is an other such immersed empty disk then there is a plane isometry $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $D' = \psi(D)$ and $\varphi = \varphi' \circ \psi$. We say that (D, φ) is the ***maxi-disk*** of the point x.

G.2 Preliminaries on the Voronoi diagram

The (1-skeleton of the) ***Voronoi diagram*** of (S, V) is the set $\mathcal{V}$ of points $x \in S$ such that the distance between x and V is realized by at least two distinct paths in S.

▶ **Lemma 34.** *The Voronoi diagram $\mathcal{V}$ of (S, V) is a graph with finitely many vertices, of minimum degree greater than or equal to three, and whose edges are geodesic segments.*

Proof. Let (D, φ) be the maxi-disk of a point $x \in S$, and let $x^\star$ be the center of D. The geodesic paths between $x^\star$ and $\varphi^{-1}(V)$ correspond via φ to the shortest paths between x and V. So $x \in \mathcal{V}$ if and only if $\varphi^{-1}(V)$ contains $m \geq 2$ points. Assume $x \in \mathcal{V}$. Let X be (the 1-skeleton of) the classical Voronoi diagram of $\varphi^{-1}(V)$ in the plane. Then X is made of m geodesic rays emanating from $x^\star$. There is an open ball $O \subset D$ on which φ is injective, containing $x^\star$, and such that $\varphi(X \cap O) = \mathcal{V} \cap \varphi(O)$. There are two cases. If $m = 2$ then $\mathcal{V}$ is locally a geodesic path around x. If $m \geq 3$ then $\mathcal{V}$ is locally a geodesic star whose central vertex is x. In particular $\mathcal{V}$ is a graph whose minimum degree is greater than or equal to three, and whose edges are geodesic segments. And $\mathcal{V}$ has finitely many vertices since S is compact. ◀

G.3 Computation of the Voronoi diagram

We compute the Voronoi diagram $\mathcal{V}$ of (S, V) by slightly extending [12, Theorem 13]. They compute the shortest paths emanating from a point $x_0 \in S$ by decomposing S according to how those paths visit the fragments of the input portalgon P. They describe a discrete process that simulates the propagation of some waves on the surface. Their waves all start from the point x_0. We adapt their strategy to simulate waves that start from all the points in V, so that the waves meet along $\mathcal{V}$. That simplifies the algorithm since waves now meet along a geodesic graph (by Lemma 34) and do not go through curved points. More precisely we prove the following:

▶ **Lemma 35** (Extension of [12, Theorem 13]). *One can compute in $O^*(n^2h)$ time a triangular portalgon P' of S with $O(n^2h)$ fragments, and a subgraph $\mathcal{V}$ of $\mathcal{C}(P')$, such that $\mathcal{V}$ is the Voronoi diagram of (S, V).*

The rest of this section is devoted to the proof of Lemma 35. We let F contain the fragments of P. Without loss of generality we assume that they are pairwise-disjoint in the plane, and we denote by ρ the projection of the union of the fragments of P onto the surface S.

Given a fragment $f \in F$, we consider immersed disks (D, φ) such that the center of D belongs to f and φ agrees with ρ on $\overline{D} \cap f$. Every immersed disk we consider is like that, without further mention. Consider the union U of the disks D over the immersed disks (D, φ). For every two immersed disks (D_0, φ_0) and (D_1, φ_1) the maps φ_0 and φ_1 agree on $\overline{D}_0 \cap \overline{D}_1$. So there is a covering map $\varphi_U : \overline{U} \to S$ that agrees with φ for every immersed disk (D, φ). Of particular interest to us is the set $V_f(\infty) := \varphi_U^{-1}(V)$. Indeed the intersection with f of the classical plane Voronoi diagram of $V_f(\infty)$ projects via ρ to the part of the Voronoi diagram of (S, V) that lies in $\rho(f)$. Thus computing the sets $V_f(\infty)$ for all $f \in F$ will immediately yield the Voronoi diagram of (S, V). One gets the following bound on their sizes:

▶ **Lemma 36.** *For every $f \in F$ at most $O(nh)$ points belong to $V_f(\infty)$.*

Proof. We call regions the following subsets of S: a vertex of $\mathcal{C}(P)$, the relative interior of an edge of $\mathcal{C}(P)$, and a (open) face of $\mathcal{C}(P)$. The regions partition S. For every shortest path p between a point $x \in S$ and the set V, record the sequence of regions intersected by p when directed from V to x. If two such paths p and p' end in $\rho(f)$ and have the same sequence then they correspond to the same point in $V_f(\infty)$. We claim that for every region R there are $O(nh)$ sequences ending with R. This claim implies the lemma. Let us prove the claim. A sequence is maximal if it is not a strict prefix of an other sequence. A sequence is critical if it is the maximal common prefix of two distinct maximal sequences. Every critical sequence ends with a face of $\mathcal{C}(P)$. For every face R' of $\mathcal{C}(P)$ there is at most one critical sequence ending with R'. Indeed every critical sequence is realized by two distinct paths. If two distinct critical sequences were to end with R', then at least two of the four associated paths would cross, and thus could be shortened, a contradiction. We proved that there are $O(n)$ critical sequences. So there are $O(n)$ maximal sequences. And every sequence contains $O(h)$ occurrences of R since P is $O(h)$-happy. That proves the claim, and the lemma. ◀

The key idea for computing those sets is to make the disks grow with time. More precisely to consider, for every $t \geq 0$, the following set $V_f(t)$ of points of $\mathbb{R}^2$. The set $V_f(0)$ contains the vertices of f that correspond to points of V. If $t > 0$ then $V_f(t)$ is the union of the sets $\varphi^{-1}(V)$ over the immersed disks (D, φ) such that the radius of D is smaller than or equal to t. As t increases new points may appear in $V_f(t)$, no point disappears, and the maximal value of $V_f(t)$ is $V_f(\infty)$. The evolution of V_f is a discrete finite process, with a finite number of ***dates*** t at which new points appear in $V_f(t)$ ($\rho(f)$ being a closed subset of S).

We now provide the algorithm. In the following, given a finite non-empty set $Y \subset \mathbb{R}^2$ and $y \in Y$, we denote by $Vor(y, Y)$ the closed cell of y in the plane Voronoi diagram of Y. The data structure maintains for every $f \in F$ a set X_f of points of $\mathbb{R}^2$ (in which we shall put the points appearing in V_f). Central to the algorithm is the notion of ***candidate event***. If $f \in F$, and if s is a side of f matched to a side s' of some $f' \in F$, then there is an orientation-preserving isometry of the plane τ that maps s to s' and puts $\tau(f)$ and f' side by side. Now assume that there is $x \in X_f$ such that $\tau(x) \notin X_{f'}$. Further assume that $Vor(x, X_f) \cap s \neq \emptyset$, and let t be the distance between x and $Vor(x, X_f) \cap s$. Then $(t, f', s', \tau(x))$ is a candidate event whose ***date*** is t. The algorithm is the following. For every $f \in F$ initialize X_f with the vertices of f corresponding to points of V, if any. Then, as long as there is a candidate event, consider any candidate event (t, f, s, x) of smallest date t, insert x in X_f, and repeat (after updating the set of candidate events). We shall detail how to compute a candidate event of smallest date, or to assert that there is no candidate event. But first we prove:

▶ **Lemma 37.** *The wave algorithm terminates. In the end $X_f = V_f(\infty)$ for every $f \in F$.*

Proof. Consider the following invariant: there is $t > 0$ such that for every $f \in F$ the set X_f contains all the points appearing in V_f at a date strictly smaller than t, and every other point of X_f appears in V_f at date t. The invariant holds after the initialization phase of the algorithm. Assume that it holds at the beginning of an iteration of the loop. This invariant implies the Property (P) that for every $f \in F$, if a point $y \in f$ is at distance $t' \geq 0$ from X_f, then the distance between $\rho(y)$ and V in S is smaller than or equal to t'; For otherwise y would be at distance greater than t' from $V_f(\infty)$, and thus at distance greater than t' from X_f since $X_f \subseteq V_f(\infty)$ by the invariant, a contradiction. So if there is a candidate event (t', f, s, x) with $t' \leq t$, then $t' = t$ and x appears in V_f at the date t. In the other direction assume that there are $f \in F$ and $x \notin X_f$ that appears at date t in V_f. We claim that there is a candidate event whose date is smaller than or equal to t (and thus equal to t by preceding). This claim implies that the invariant holds at the end of the iteration, and that the algorithm does not stop until $X_f = V_f(\infty)$ for all $f \in F$, which proves the lemma.

Let us now prove the claim. There are a point y on the boundary of f and an immersed disk (D, φ) centered at y, of radius t, such that $x \in \varphi^{-1}(V)$. There are two cases. First assume that y lies in the relative interior of a side s of f. Then s is matched to a side s' of f' for some $f' \in F$. Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the orientation-preserving isometry that maps s to s' and puts $\tau(f)$ and f' side by side. Then $\tau(x)$ appears in $V_{f'}$ at a date $t' < t$. And so $\tau(x) \in X_{f'}$ by our invariant. Moreover $\tau(y)$ is at distance greater than or equal to t from $X_{f'}$ by Property (P). So the distance between $\tau(y)$ and $X_{f'}$ is t, the distance between $\tau(y)$ and $\tau(x)$. We proved that in the plane Voronoi diagram of $X_{f'}$ the closed cell of $\tau(x)$ intersects s' in $\tau(y)$ (at least), which implies the claim in this case.

Now assume that y is a vertex of f. Then $\rho(y)$ is a vertex of $\mathcal{C}(P)$, and it is a flat point of S since $t > 0$. Consider the $k \geq 2$ directed edges emanating from $\rho(y)$ in $\mathcal{C}(P)$, and lift them in the plane by straight line segments $s_0, \ldots, s_{k-1}$ emanating from y. For every i consider the corner α between s_i and s_{i+1}, indices are modulo k. Then α corresponds to a corner β of some $f \in F$. Let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the plane isometry that maps β to α, and let $X_i := \tau(X_f)$. There is i such that $x \in X_i$ and $x \notin X_{i+1}$. The distance between y and X_i is greater than or equal to t by Property (P), so it is equal to t, the distance between y and x. We proved that in the plane Voronoi diagram of X_i the closed cell of x intersects the segment s_i in y (at least), which implies the claim, and the lemma. ◀

All there remains to do is to detail how to compute a candidate event of smallest date, or to assert that there is no candidate event. That can be done naively in time polynomial in n and h. However, to gain efficiency, we shall maintain the list of candidate events sorted by date (implemented with a balanced search tree). For that we consider the following setting. Consider a closed segment of positive length s in $\mathbb{R}^2$ (such as the side of some $f \in F$) and a list of $k \geq 1$ points $z_1, \ldots, z_k \in \mathbb{R}^2$ (such as the points to be inserted in X_f in order of insertion). For every $0 \leq i \leq k$ let $Z_i = \{z_0, \ldots, z_i\}$. We shall maintain in an online manner after each insertion of a point z_i, $1 \leq i \leq k$, the sets $Vor(y, Z_i) \cap s$, $y \in Z_i$. For that let U_i contain the points $y \in Z_{i-1}$ that require an update when inserting z_i, equivalently, such that $Vor(y, Z_{i-1}) \cap s \neq Vor(y, Z_i) \cap s$. Then:

▶ **Lemma 38.** *The sum over $1 \leq i \leq k$ of the cardinality of the set U_i is smaller than or equal to $5k$.*

Proof. Let $1 \leq i \leq k$. We prove the lemma by proving that at most four points $y \in U_i$ are such that $Vor(y, Z_i) \cap s \neq \emptyset$. Let $s' \subseteq s$ contain the points strictly closer to z_i than to Z_{i-1}. Then $Vor(y, Z_{i-1}) \cap s$ contains a point in s' and a point not in s'. Also $Vor(y, Z_{i-1})$ and s' are intervals, and s' is open in the topology of s (note that s' may contain endpoints of s). So there is an arbitrarily small open interval $s'' \subset Vor(y, Z_{i-1}) \cap s'$ that shares one of its endpoints with s'. There at most two points $y \in Z_{i-1}$ such that $s'' \subset Vor(y, Z_{i-1})$, and that there are two such ends s'' of s'. That proves the lemma. ◀

▶ **Lemma 39.** *There is an online algorithm to which we give the points $z_1, \ldots, z_k$ in this order, that after receiving z_i, $1 \leq i \leq k$, returns U_i and $Vor(y, Z_i) \cap s$ for all $y \in U_i \cup \{z_i\}$, and runs in $O(k \log k)$ total time.*

Proof. Let $1 \leq i \leq k$. There is a partition of s into points an open intervals such that for every $y \in Z_{i-1}$ the set $Vor(y, Z_{i-1}) \cap s$ is the closure of one of the partition sets. We maintain the list of tuples $(y, Vor(y, Z_{i-1}) \cap s)$ over $y \in Z_{i-1}$ ordered by the position of $Vor(y, Z_{i-1}) \cap s$ along s (after directing s arbitrarily, and ordering arbitrarily any two points $y \neq y' \in Z_{i-1}$ for which $Vor(y, Z_{i-1}) \cap s = Vor(y', Z_{i-1}) \cap s$). Given such a tuple $(y, Vor(y, Z_{i-1}) \cap s)$ we can determine in constant time whether $y \in U_i$ by checking whether there is a point of $Vor(y, Z_{i-1}) \cap s$ strictly closer to z_i than to y. If $y \notin U_i$, then we can, again in constant time, either correctly assert that all the tuples $(y', Vor(y', Z_{i-1}) \cap s)$ before $(y, Vor(y, Z_{i-1}) \cap s)$ in the list are such that $y' \notin U_i$, or correctly correctly assert that all the tuples after $(y, Vor(y, Z_{i-1}) \cap s)$ are like that. So we can list by dichotomy the $k' \geq 0$ tuples $(y, Vor(y, Z_{i-1}) \cap s)$ such that $y \in U_i$ in $O(k' + \log k)$ time. For every $y \in U_i$ we compute $Vor(y, Z_i) \cap s$, and we update the list of tuples accordingly, in $O(\log k)$ time per point, so in $O(k' \log k)$ total time. We compute $Vor(z_i, Z_i) \cap s$ in $O(\log k)$ time by finding by dichotomy the first and last tuples $(y, Vor(y, Z_{i-1}) \cap s)$ such that $Vor(y, Z_{i-1}) \cap s$ contains a point whose distance to y is greater than or equal to its distance to z_i, if any. That proves the lemma. ◀

Proof of Lemma 35. The wave algorithm never inserts a point in a set X_f, $f \in F$, that was already there before. So the algorithm terminates after $O(n^2 h)$ insertions by Lemma 36 and Lemma 37. In the end the plane Voronoi diagram of X_f projects via ρ to the part of the Voronoi diagram of (S, V) that lies in $\rho(f)$. All those diagrams can be computed in $O^*(n^2 h)$ total time with classical algorithms. Cutting the fragments of P along those diagrams, and triangulating the fragments that are not triangles, provides the desired portalgon P'. To compute a candidate event of smallest date, or to correctly determine that there is no candidate event, we maintain the list of candidate events sorted by date. We update this list when inserting a point x in a set X_f, in amortized time $O(\log(nh))$ per insertion, as follows.

First, the $k \geq 0$ candidate events of the form $(\cdot, f, \cdot, x)$ must all be deleted. This is done in time $O((k+1)\log(nh))$ by maintaining a second list of the candidate events sorted by the position of their point (lexicographic order on its coordinates, say). All but $O(\log(nh))$ of the time spent is amortized by the fact that every event deleted here was created earlier in the execution of the algorithm. Second, consider a side s of f, matched to a side s' of some $f' \in F$, and let $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ be the orientation-preserving isometry that maps s to s' and puts $\tau(f)$ and f' side by side. Among the candidate events of the form $(\cdot, f', s', \tau(y))$, $y \in X_f$, those for which $Vor(y, X_f \cup \{x\}) \cap s \neq Vor(y, X_f) \cap s$ may have to updated. If $Vor(y, X_f \cup \{x\}) \cap s = \emptyset$, then the event must be deleted. Otherwise, only the date of the event may change. This is done with Lemma 39. The lemma also provides us with the set $Vor(x, X_f \cup \{x\}) \cap s$. If this set is not empty, and if $\tau(x) \notin X_{f'}$, then we create the corresponding event, and we insert it in our lists in $O(\log nh)$ time. ◀

G.4 The duality between the Delaunay tessellation and the Voronoi diagram

We now describe the duality between the Delaunay portalgon $\mathcal{D}$ and the Voronoi diagram $\mathcal{V}$ of (S, V). Let (D, φ) be the maxi-disk of a point $x \in S$. If the convex hull of $\varphi^{-1}(V)$ projects via φ to the closure of a face f of $\mathcal{D}$, then we say that x is ***dual*** to f.

▶ **Lemma 40.** *The duality relation is a one-to-one correspondence between the vertices of $\mathcal{V}$ and the faces of $\mathcal{D}$.*

Proof. Let $x \in S$. Let (D, φ) be the maxi-disk of x, and let m be the number of points in $\varphi^{-1}(V)$. The convex hull of $\varphi^{-1}(V)$ projects via φ to the closure of a face of $\mathcal{D}$ if and only if $m \geq 3$. And we already proved that $m \geq 3$ if and only if x is a vertex of $\mathcal{V}$. Every face of $\mathcal{D}$ can be obtained from a vertex of $\mathcal{V}$ in this way by definition of the Delaunay tessellation. And distinct vertices of $\mathcal{V}$ project to distinct faces of $\mathcal{D}$ for otherwise they would have the same maxi-disk. ◀

Let v be a vertex of $\mathcal{V}$, dual to a face f of $\mathcal{D}$. We call ***side*** of f any directed edge of $\mathcal{D}$ that sees f on its left. We now relate the directed edges emanating from v to the sides of f. Let (D, φ) be the maxi-disk of v. Let $v^\star$ be the center of D, and let $y_0, \ldots, y_{m-1}$ be the $m \geq 3$ points of $\varphi^{-1}(V)$. In the plane the classical Voronoi diagram of $\varphi^{-1}(V)$ is made of m geodesic rays $r_0, \ldots, r_{m-1}$ emanating from $v^\star$, so that $r_0, y_0, \ldots, r_{m-1}, y_{m-1}$ are in clockwise order around $v^\star$. There is an open ball $O \subset D$ on which φ is injective, containing $v^\star$, such that within O the rays $r_0, \ldots, r_{m-1}$ correspond via φ to the directed edges $e_0, \ldots, e_{m-1}$ emanating from v in $\mathcal{V}$. For every i the geodesic path from y_i to y_{i+1} corresponds via φ to a side e'_i of f, indices are modulo m. We say that e_i and e'_i are ***dual***. This duality is a one-to-one correspondence that maps the cyclic order of directed edges emanating from v around v to the cyclic order of sides of f along the boundary of f.

▶ **Lemma 41.** *If a directed edge e_0 of $\mathcal{V}$ is dual to a directed edge e'_0 of $\mathcal{D}$, then the reversal of e_0 is dual to the reversal of e'_0.*

Proof. Let e'_1 be the reversal of e'_0, and let e_1 be the dual of e'_1. We shall prove that e_1 is the reversal of e_0. Consider the maxi-disks (D_0, φ_0) and (D_1, φ_1) of the base-vertices of e_0

and e_1, and realize them so that they agree on the geodesic segment p that is the pre-image of the common edge of e'_0 and e'_1. Then φ_0 and φ_1 agree on $\overline{D}_0 \cap \overline{D}_1$, so they agree with a common map $\varphi_0 \cup \varphi_1 : \overline{D}_0 \cup \overline{D}_1 \to S$. Let q be the geodesic segment between the centers of D_0 and D_1. Then q is contained in $\overline{D}_0 \cup \overline{D}_1$, and projects via $\varphi_0 \cup \varphi_1$ to the common edge of e_0 and e_1 in $\mathcal{V}$. Indeed for every point $x^\star$ in the relative interior of q the maxi-disk (D, φ) of $\varphi(x^\star)$ can be realized so that $x^\star$ is the center of D, and so that φ agrees with $\varphi_0 \cup \varphi_1$ on $\overline{D} \cap (\overline{D}_0 \cup \overline{D}_1)$. Then $\varphi^{-1}(V)$ contains exactly the two endpoints of p, and so $\varphi(x^\star)$ belongs to the relative interior of an edge of $\mathcal{V}$. ◀

Note that the vertices of $\mathcal{V}$ do not necessarily belong to their dual faces in $\mathcal{D}$, and that dual edges do not necessarily cross.

G.5 Computing the canonical portalgon: proof of Proposition 20

In this section we prove Theorem 20. As a preliminary we need a definition and a lemma. Let W be a walk in the dual of some triangulation M. To ease the reading assume that in M every edge is incident to two distinct faces. The following definition extends in a straightforward manner to general triangulations. In the plane realize the $k \geq 1$ faces visited by W isometrically, and respecting their orientation, by respective triangles $U_1, \ldots, U_k$. Make sure that for every $1 \leq i < k$ the triangles U_i and U_{i+1} agree on the placement of the i-th edge of M crossed by W. The resulting sequence $U = (U_1, \ldots, U_k)$ is an ***unfolding*** of W. In general a vertex of M may have several occurrences among the vertices of the triangles in U, and those occurrences may be at distinct points in the plane. Yet:

▶ **Lemma 42.** *Let $\mathcal{V}$ be the Voronoi diagram of (S, V). Let F be a face of $\mathcal{V}$. There is a unique point $w \in F \cap V$. Let U be an unfolding of a walk in the dual of some triangulation of F. In U all occurrences of w are at the same point.*

Proof. Our first claim is that F is simply connected, and that $F \cap V$ contains a single point w. To prove the claim first consider a point $x \in F$. There is a unique shortest path p from x to V. Then p is disjoint from $\mathcal{V}$. So the endpoint of p belongs to F. That proves $F \cap V \neq \emptyset$. Now consider the universal covering space $\widetilde{F}$ of F. Then $\widetilde{F}$ does not contain two distinct lifts of points of V. For otherwise let $\widetilde{V}$ contain the lifts of the points of V in $\widetilde{F}$. There is a point $\widetilde{x} \in \widetilde{F}$ whose distance to $\widetilde{V}$ is realized by two distinct paths. And $\widetilde{x}$ lifts a point of $\mathcal{V}$, a contradiction. That proves the first claim.

Our second claim is that around any vertex v of $\mathcal{V}$ the angles between consecutive edges are all smaller than or equal to π. Indeed let (D, φ) be the maxi-disk of v. Let $v^\star$ be the center of D. Let X be the Voronoi diagram of $\varphi^{-1}(V)$ in the plane. The faces of X are all convex, being intersections of half-planes. So the angles between consecutive rays of X around $v^\star$ are all smaller than or equal to π. There is an open disk O on which φ is injective, containing $v^\star$, such that $\varphi(X \cap O) = \mathcal{V} \cap \varphi(O)$. That proves the second claim.

The first claim implies that F is homeomorphic to an open disk since F is not homeomorphic to a sphere. It also implies that F has no curved point except possibly w since V contains all curved points of S. Let $\widehat{F}$ be the surface homeomorphic to a closed disk obtained by cutting the closure of F along the boundary of F. The second claim implies that the angles at the corners of $\widehat{F}$ are smaller than or equal to π. So the shortest paths between those corners and w are, together with the boundary edges of $\widehat{F}$, the edges of a triangulation N of $\widehat{F}$. The dual of N is a cycle, and w is the central vertex of N. If U is an unfolding of a dual walk of N, then all occurrences of w in U are at the same point in the plane. That easily extends to every other triangulation M of $\widehat{F}$ from the fact that there is a triangulation

R that is a refinement of both M and N, in the sense that the 1-skeleton of R contains subdivisions of the 1-skeletons of M and N. ◀

Proof of Theorem 20. Apply Lemma 35 to replace P in $O^*(n^2h)$ time by a triangular portalgon P' of S with $O(n^2h)$ fragments, and to compute a subgraph $\mathcal{V}$ of $\mathcal{C}(P')$ such that $\mathcal{V}$ is the Voronoi diagram of (S, V). Cutting S along the Delaunay tessellation of (S, V) gives the canonical portalgon Q. We now describe how to compute Q from $\mathcal{V}$.

First we build the combinatorics of Q as follows. For every vertex v of $\mathcal{V}$ create a fragment f in Q whose number of sides equals the degree of v. Identify the sides of f with the directed edges of $\mathcal{V}$ emanating from v, in order. For every edge e of $\mathcal{V}$ the two directions of e correspond to two distinct sides of fragments of Q (possibly of the same fragment), identify those two sides. This is correct by Lemma 40 and Lemma 41.

There remains to show how to realize in the plane the face f of Q dual to a given vertex v of $\mathcal{V}$, and to identify the $m \geq 3$ sides of f to the directed edges $e_0, \ldots, e_{m-1}$ emanating from v in $\mathcal{V}$. Recall that in the maxi-disk (D, φ) of v the points of $\varphi^{-1}(V)$ can be listed as $y_0, \ldots, y_{m-1}$ so that for every i the geodesic segment from y_i to y_{i+1} projects to the side of f dual to e_i, indices are modulo m. The issue is that we do not have access to (D, φ). Yet we can compute the points $y_0, \ldots, y_{m-1}$ as follows. Realize v by an arbitrary point $v^\star$ in the plane, and realize the faces of $\mathcal{C}(P')$ incident to v isometrically around $v^\star$ in the plane. This is possible since v is flat. Without loss of generality the center of D is $v^\star$, and φ agrees with the realization of the faces around $v^\star$. Let $F_0, \ldots, F_{m-1}$ be the faces of $\mathcal{V}$ occurring around v, so that each face F_i is in-between the directed edges e_i and e_{i+1} around v. Note that each face of $\mathcal{V}$ may contain several faces of $\mathcal{C}(P')$, and may occur several times around v. For every i the face F_i contains a single point $w_i \in V$ by Lemma 42. Consider a walk W in the dual of $\mathcal{C}(P') \cap F_i$ that starts with a face W_0 incident to v, and visits a face incident to w_i. Unfold the faces visited by W in the plane, starting from the realization of W_0 around $v^\star$. Let $w_i^\star$ be some arbitrary occurrence of w_i in the unfolding. Then $w_i^\star$ does not depend on W nor on the choice of the ocurrence of w_i in the unfolding by Lemma 42. We claim that $w_i^\star = y_i$. To prove this claim we show that y_i can also be obtained as an occurence of w_i in such an unfolding of a walk in the dual of $\mathcal{C}(P') \cap F_i$. Indeed the geodesic path p from $v^\star$ to y_i projects via φ to a shortest path from v to V. And $\varphi \circ p$ immediately enters F_i after leaving v. So $\varphi \circ p$ is relatively included in F_i, and thus ends at w_i. By slightly perturbing p without changing its endpoints we may ensure that $\varphi \circ p$ corresponds to a walk in the dual of $\mathcal{C}(P') \cap F_i$, which is as desired. That proves the claim.

Achieving the claimed running time requires a last technicality. Consider a face F of $\mathcal{V}$, containing a point $w \in V$. Recall that for some faces W_0 of $\mathcal{C}(P') \cap F$ we need to construct a dual walk W from W_0 to w, unfold W, and retain the relative positions of some occurences of W_0 and w in the unfolding. Doing so independently for every face W_0 of $\mathcal{C}(P') \cap F$ may take too long as we would visit faces of $\mathcal{C}(P')$ several times. Instead we consider a single spanning tree Y in the dual of $\mathcal{C}(P') \cap F$, we unfold the faces of $\mathcal{C}(P') \cap F$ along Y, and we retrieve all the required information from the unfolding. (Note that the choice of Y does not matter, and that the unfolding may overlap). Doing that in every face F of $\mathcal{V}$ takes $O(n^2h)$ time in total since P' has $O(n^2h)$ fragments. ◀